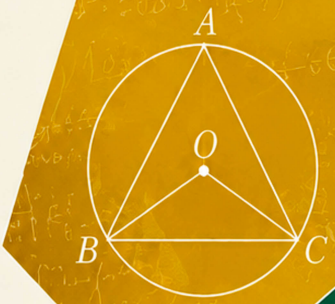
COLLECTION OF

PROBLEMS

IN

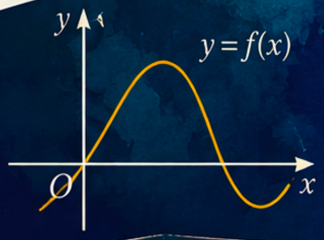
HIGHER MATHEMATICS



*** MADHAVA MATHEMATICS COMPETITION***

(MMC)

(A Mathematics Competition for
Indian Undergraduate Students)



$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

PREFACE

Collection of Problems in Higher Mathematics — MADHAVA MATHEMATICS COMPETITION (MMC): A Mathematics Competition for Indian Undergraduate Students (2010–2026)

The Madhava Mathematics Competition (MMC) occupies a distinctive place in the mathematical life of Indian undergraduate students. Conceived as a competition that encourages students to look beyond routine university exercises, it provides a setting in which mathematical knowledge is combined with creativity, persistence, and rigorous proof. This collection brings together problems from 2010–2026, presenting a sustained record of mathematical challenges offered to successive generations of undergraduate contestants.

The title of the competition recalls Madhava of Sangamagrama, one of the great mathematicians of the Kerala school of mathematics. The name is especially appropriate for a competition devoted to higher mathematics: it evokes a long Indian tradition of mathematical investigation while placing modern students in a contemporary problem-solving environment. The problems in this collection likewise move between classical ideas and modern methods, requiring contestants to recognize structure rather than merely apply memorized procedures.

A university mathematics competition differs fundamentally from a conventional examination. In an examination, a student may often know which theorem or algorithm is expected. In a competition problem, the principal difficulty may be discovering what should be tried in the first place. A successful contestant must experiment, formulate conjectures, identify useful invariants, recognize hidden connections, and ultimately convert an intuitive idea into a logically complete argument.

The MMC problems collected here reflect this philosophy. They draw on areas such as algebra, number theory, geometry, combinatorics, inequalities, calculus, linear algebra, and related branches of higher mathematics. Some problems have short and elegant solutions; others require a sustained chain of ideas. In both cases, the essential lesson is the same: mathematical sophistication often lies not in using complicated machinery, but in finding the right point of view.

The chronological span from 2010 to 2026 makes the collection useful both as a problem book and as a record of undergraduate mathematical competition in India. Read year by year, the problems reveal recurring themes and techniques while also displaying changes in style and emphasis. Read independently, they form a varied training ground in which a student can encounter unfamiliar ideas and learn to work without the guidance of a prescribed method.

For students, the most valuable way to use this collection is to attempt each problem before reading a solution. Drawing figures, testing small cases, searching for patterns, considering extreme cases, and trying apparently unsuccessful approaches are all part of genuine mathematical work. A failed attempt is rarely wasted: it can reveal what the problem is not about and point toward the structure that a successful solution must exploit.

For teachers and mentors, the collection offers material for problem sessions, enrichment courses, mathematical clubs, and advanced undergraduate training. A single problem can support several levels of discussion: discovery of the main idea, construction of a rigorous proof, comparison of alternative solutions, and possible generalizations. Such discussion is an important part of learning how mathematicians think and communicate.

The wider significance of a competition such as MMC extends beyond prizes and rankings. Mathematical problem solving cultivates habits of mind valuable throughout science and technology: careful reasoning, persistence in the face of difficulty, attention to definitions, skepticism toward unsupported claims, and the ability to simplify a complicated situation without losing its essential structure. These habits remain useful long after a particular contest has ended.

This collection is therefore intended as more than an archive. It is an invitation to explore higher mathematics through problems that demand thought. Each question offers an opportunity to discover a connection that may not be visible at first sight, to experience the satisfaction of a successful argument, and sometimes to learn even more from an unsuccessful approach. Taken together, the problems form a portrait of mathematical creativity among Indian undergraduates over the years 2010–2026.

May this collection serve students preparing for mathematical competitions, teachers who wish to inspire deeper engagement with mathematics, and readers who simply enjoy challenging problems. Above all, may it encourage the spirit at the heart of mathematical discovery: curiosity, patience, clarity, and the willingness to keep asking how a difficult question can be made understandable.

MADHAVA MATHEMATICS COMPETITION

Organized by
Department of Mathematics, S. P. College, Pune
Funded by
National Board for Higher Mathematics

Date : 3 / 1 / 2010 **Marks : 100**

Time : 12.00 noon to 3.00 p.m.

N. B.

1) All questions are compulsory.

2) Part I carries 20 marks, Part II carries 20 marks and Part III carries 60 marks.

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Part I

N.B. Each question in part I carries 2 marks.

1. The number of subsets of the set $\{1, 2, \cdots, 10\}$ containing at least one odd integer is
   (a) 2^{10} (b) 2^5 (c) ${}^{10}C_5$ (d) $2^{10} - 2^5$.

2. $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2009)^2 - (2010)^2$ is equal to
   (a) zero
   (b) -2021055
   (c) -2019045
   (d) -1010555.

3. The coefficient of $(x-1)^3$ in the Taylor expansion of $(x-1)^3 \cos(\pi x)$ about $x = 1$ is
   (a) -1 (b) 1 (c) 6 (d) -6.

4. The number of non-zero solutions of $z^2 + 2\overline{z} = 0$ is
   (a) 2 (b) 3 (c) 4 (d) 5.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 5x - 6 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   Then f is continuous at
   (a) no real number (b) -2 and -3 (c) all rationals (d) 2 and 3.

6. The number of negative solutions of the equation $e^x - \sin x = 0$ is
   (a) 1 (b) 2 (c) 0 (d) infinite.

7. Let $a_n = \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$ and $b_n = \left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n$, and let T_n denote the area of the triangle with sides $a_n - 1, a_n, a_n + 1$. Then
   (a) a_n, b_n, T_n are all integers for each $n \in \mathbb{N}$.
   (b) a_n, T_n are all integers for each $n \in \mathbb{N}$.
   (c) T_n is not an integer for each $n \in \mathbb{N}$.
   (d) a_n is an integer for even n and b_n is an integer for odd n.

8. Let $A = (a_{ij})$ be an $m \times n$ matrix where

$$a_{ij} = \begin{cases} 0 & \text{if } i+j \text{ is even} \\ 1 & \text{if } i+j \text{ is odd.} \end{cases}$$

   Then the rank of $A = (a_{ij})$ is
   (a) m (b) n (c) 2 (d) 3.

9. The last two digits of 17^{400} are
   (a) 17 (b) 09 (c) 01 (d) 89.

10. The number of values of a for which the equation $x^3 - x + a = 0$ has a double root is
    (a) 0 (b) 1 (c) 2 (d) infinite.

Part II

N.B. Each question in part II carries 5 marks.

1. Find the remainder when $f(x^3)$ is divided by $f(x)$ where $f(x) = 1 + x + x^2$.

2. Let $f : [0,1] \to [0,1]$ be a function defined as follows:
   $f(1) = 1$, and if $a = 0.a_1a_2a_3....$ is the decimal representation of a (which does not end with a chain of 9's), then $f(a) = 0.0a_10a_20a_3.....$
   Discuss the continuity of f at 0.392.

3. A bubble chamber contains 3 types of particles. 100 of type x, 200 of type y and 300 of type z. Whenever x and y particles collide they both become z particles, likewise when y and z collide they both become x, and when x and z collide they both become y.
   Can the particles in the chamber evolve so that there remain particles of only one type?

4. Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that f is onto, g is one-one and $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. Prove that $f = g$.

Part III

N.B. Each question in part III carries 12 marks.

1. Abhi and Ash play the following game:
   A blank 2010×2010 array is taken. Abhi starts the game by writing a real number in any one of the squares of the array. Then Ash writes a real number in any blank square of the array. The game is continued till all the squares are filled with numbers. Abhi wins the game if the determinant of the resulting matrix is non-zero and Ash wins the game if the determinant of the resulting matrix is 0.
   Show that Ash can always win the game.

2. Show that the polynomial equation with real coefficients $a_nx^n + a_{n-1}x^{n-1} + ... + a_3x^3 + x^2 + x + 1 = 0$ cannot have all real roots.

3. Find the sum $\sum_{j=0}^{n} \sum_{i=j}^{n} {}^nC_i \, {}^iC_j$.

4. Find the g.c.d. of the numbers $\{2^{13} - 2, 3^{13} - 3, 4^{13} - 4, \cdots, 13^{13} - 13\}$.

5. Let $\{a_n\}$ be a sequence of real numbers. Suppose $\{a_{sn}\}$ converges for every fixed positive integer $s > 1$.
   1) If $a_{sn} \to a$ and $a_{tn} \to b$ for some fixed positive integers s and t, then is $a = b$? Justify.
   2) Is the sequence $\{a_n\}$ convergent? Justify.

Part I

N.B. Each question in part I carries 2 marks.

1. The number of subsets of the set $\{1, 2, \cdots, 10\}$ containing at least one odd integer is
   (a) 2^{10} (b) 2^5 (c) ${}^{10}C_5$ (d) $2^{10} - 2^5$.
   Solution : (d)
   Total number of subsets of the set $\{1, 2, \cdots, 10\}$ is 2^{10}. The number of subsets of the set $\{1, 2, \cdots, 10\}$ containing only even integers is 2^5. Thus the required number is $2^{10} - 2^5$.

2. $1^2 - 2^2 + 3^2 - 4^2 + ... + (2009)^2 - (2010)^2$ is equal to
   (a) zero
   (b) -2021055
   (c) -2019045
   (d) -1010555.
   Solution : (b)
   $1^2 - 2^2 + 3^2 - 4^2 + ... + (2009)^2 - (2010)^2$
   $= (1^2 - 2^2) + (3^2 - 4^2) + ... + ((2009)^2 - (2010)^2)$
   $= (1 - 2)(1 + 2) + (3 - 4)(3 + 4) + ... + (2009 - 2010)(2009 + 2010)$
   $= (-1)(1 + 2 + 3 + 4 + ... + 2009 + 2010)$
   $= -\dfrac{(2010)(2011)}{2} = -2021055.$

3. The coefficient of $(x-1)^3$ in the Taylor expansion of $(x-1)^3 \cos(\pi x)$ about $x = 1$ is
   (a) -1 (b) 1 (c) 6 (d) -6.
   Solution : (a)
   The coefficient of $(x-1)^3$ in the Taylor expansion of $(x-1)^3 \cos(\pi x)$ about $x = 1$ is nothing but the constant term in the Taylor expansion of $\cos(\pi x)$ about $x = 1$. This constant term is $\cos \pi = -1$.

4. The number of non-zero solutions of $z^2 + 2\overline{z} = 0$ is
   (a) 2 (b) 3 (c) 4 (d) 5.
   Solution : (b)
   We have $|z^2| = |-2\overline{z}| = 2|z|$. Suppose $z \neq 0$. Then $|z|^2 = 4 = z\overline{z}$. Hence the equation becomes $z^2 + 2\dfrac{4}{z} = 0$ i.e. $z^3 + 8 = 0$. Hence there are 3 non zero solutions.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 5x - 6 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then f is continuous at
(a) no real number (b) -2 and -3 (c) all rationals (d) 2 and 3.
Solution : (d)
Let a be any real number. Suppose f is continuous at a. Let $\{a_n\}$ and $\{b_n\}$ be sequences of rationals and irrationals respectively converging to a. Then by continuity of f at a,
$\lim f(a_n) = \lim f(b_n)$
$\lim(a_n^2) = \lim(5b_n - 6)$. Therefore $a^2 = 5a - 6$. Hence $a = 2, 3$. Further it is clear that f is continuous at $2, 3$.

6. The number of negative solutions of the equation $e^x - \sin x = 0$ is
(a) 1 (b) 2 (c) 0 (d) infinite.
Solution : (d)
Graphs of e^x and $\sin x$ intersect infinitely many times for negative real numbers.

7. Let $a_n = \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$ and $b_n = \left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n$, and let T_n denote the area of the triangle with sides $a_n - 1, a_n, a_n + 1$. Then
(a) a_n, b_n, T_n are all integers for each $n \in \mathbb{N}$.
(b) a_n, T_n are all integers for each $n \in \mathbb{N}$.
(c) T_n is not an integer for each $n \in \mathbb{N}$.
(d) a_n is an integer for even n and b_n is an integer for odd n.
Solution : (b)
Here we arrive at the answer by the process of elimination ! For $n = 1$, $a_1 = 4, b_1 = 2\sqrt{3}$ and $T_1 = 6$. Therefore (a),(c) and (d) are ruled out.

8. Let $A = (a_{ij})$ be an $m \times n$ matrix where

$$a_{ij} = \begin{cases} 0 & \text{if } i + j \text{ is even} \\ 1 & \text{if } i + j \text{ is odd.} \end{cases}$$

Then the rank of $A = (a_{ij})$ is
(a) m (b) n (c) 2 (d) 3.
Solution : (c)
Observe that alternate rows are identical.

9. The last two digits of 17^{400} are
(a) 17 (b) 09 (c) 01 (d) 89.

Solution : (c)
By Euler's theorem $17^{40} \equiv 1 (\text{mod} 100)$. Therefore $17^{400} \equiv 1 (\text{mod} 100)$.

10. The number of values of a for which the equation $x^3 - x + a = 0$ has a double root is
(a) 0 (b) 1 (c) 2 (d) infinite.
Solution : (c)
Observe that the equation $f(x) = 0$ has a double root if and only if it is a common root of $f(x) = 0$ and $f'(x) = 0$. Now $f'(x) = 3x^2 - 1 = 0$ at $x = \pm\frac{1}{\sqrt{3}}$. For these values of x, we have two different values of a.

Part II

N.B. Each question in part II carries 5 marks.

1. Find the remainder when $f(x^3)$ is divided by $f(x)$ where $f(x) = 1 + x + x^2$.
Solution : Let $f(x) = 1+x+x^2$. Then $(x-1)f(x) = x^3-1$. Therefore $f(x)|(x^3 - 1)$.
Now $f(x^3) = 1 + x^3 + x^6 = 3 - 2 + x^3 + x^6 = (x^3 - 1) + (x^6 - 1) + 3$. Therefore $f(x)|[f(x^3) - 3]$. Hence the required remainder is 3, since f being a polynomial of degree 2, the unique remainder is of the form $ax + b$.

2. Let $f : [0, 1] \to [0, 1]$ be a function defined as follows:
$f(1) = 1$, and if $a = 0.a_1a_2a_3....$ is the decimal representation of a (which does not end with a chain of 9's), then $f(a) = 0.0a_10a_20a_3.....$
Discuss the continuity of f at 0.392.
Solution : Let $a = 0.392$. Thus $f(a) = 0.030902$. We shall prove that the given function is not continuous at a. We construct a sequence x_n converging to a such that $f(x_n)$ does not converge to $f(a)$. Let $x_n = 0.39199\ldots9$, where 9 occurs n times at the end. Then $x_n \to a$.
Now $f(x_n) = 0.03090109090\ldots09$, where 09 occurs n times at the end. Observe that $f(x_n)$ does not converge to $f(a) = 0.030902$.

3. A bubble chamber contains 3 types of particles. 100 of type x, 200 of type y and 300 of type z. Whenever x and y particles collide they both become z particles, likewise when y and z collide they both become x, and when x and z collide they both become y.
Can the particles in the chamber evolve so that there remain particles of only one type?
Solution : Suppose r, s, t are the number of x, y and z particles in the beginning. When x and y collide we get $(r - 1, s - 1, t + 2)$ as the new numbers say (r', s', t'). We notice that
$t - r \equiv t' - r' (\text{mod } 3)$.

Same is true about $r-s$ and $s-t$.

Now in the beginning $r = 100, s = 200, t = 300$ Hence $r - s \equiv 2 \pmod 3$.

$s - t \equiv 2 \pmod 3$.

$t - r \equiv 2 \pmod 3$.

Thus at any stage $r \not\equiv s \pmod 3$,$s \not\equiv t \pmod 3$,$t \not\equiv r \pmod 3$. But if the particles end up in only one type, then two of r, s, t become zero, say $\overline{r} = \overline{s} = 0$. Then certainly $\overline{r} \equiv \overline{s} (\text{mod} 3)$. This is not possible.Hence it is impossible for the particles to end up in one type only.

4. Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that f is onto, g is one-one and $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. Prove that $f = g$.

   Solution : We shall prove the statement $P(k)$: For every $k \in \mathbf{N}$, there exists a unique $x_k \in \mathbf{N}$ such that $f(x_k) = g(x_k) = k$ by induction on k. Given that f is onto. Thus there exists $x_1 \in \mathbf{N}$ such that $f(x_1) = 1$. But $g \leq f$, so $g(x_1) = 1$. Since g is one-one this x_1 is unique. Thus we have proved $P(1)$. Now let $P(k)$ be true. We shall prove that $P(k+1)$ is true. As f is onto, there exists $x_{k+1} \in \mathbf{N}$ such that $f(x_{k+1}) = k+1$. But $g \leq f$ and by induction hypothesis g already takes all values less that $k+1$. So $g(x_{k+1}) = k+1$. Since g is one-one this x_{k+1} is unique. Thus by the principle of mathematical induction, the $P(k)$ holds for all natural numbers k. Observe that $P(k)$ implies $f = g$.

Part III

N.B. Each question in part III carries 12 marks.

1. Abhi and Ash play the following game:

   A blank 2010×2010 array is taken. Abhi starts the game by writing a real number in any one of the squares of the array. Then Ash writes a real number in any blank square of the array. The game is continued till all the squares are filled with numbers. Abhi wins the game if the determinant of the resulting matrix is non-zero and Ash wins the game if the determinant of the resulting matrix is 0.

   Show that Ash can always win the game.

   Solution : First note that 2010^2 is an even number. So, since Abhi starts the game, Ash always enters the last number in the array. Hence she tries to make two rows identical. In particular first two.

2. Show that the polynomial equation with real coefficients $a_n x^n + a_{n-1} x^{n-1} + ... + a_3 x^3 + x^2 + x + 1 = 0$ cannot have all real roots.

   Solution : Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_3 x^3 + x^2 + x + 1$. Note that $p(0) \neq 0$. Thus it is sufficient to prove that $q(x) = p(\frac{1}{x}) = 0$ cannot have all real roots. Now ,

$$q(x) = x^n + x^{n-1} + x^{n-2} + \cdots + a_n$$

Let $b_1, b_2, \ldots, b_n$ be the roots of $q(x) = 0$. Then $\sum_{i=1}^{n} b_i = -1$ and $\sum b_i b_j = 1$. Thus

$$\sum b_i^2 = (\sum b_i)^2 - 2(\sum b_i b_j) = 1 - 2(1) = -1$$

But if all b_is are real, then $\sum b_i^2 > 0$. Thus all the b_is cannot be real.

3. Find the sum $\sum_{j=0}^{n} \sum_{i=j}^{n} {}^nC_i \, {}^iC_j$.
Solution : $\sum_{j=0}^{n}\sum_{i=j}^{n}\binom{n}{i}\binom{i}{j}$
$=\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}$
$=\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{i}\binom{i}{j}$
$=\sum_{i=0}^{n}\binom{n}{i}2^i$
$= 3^n$.

4. Find the g.c.d. of the numbers $\{2^{13}-2, 3^{13}-3, 4^{13}-4, \cdots, 13^{13}-13\}$.
Solution : Let d be gcd of the numbers $\{2^{13}-2, 3^{13}-3, 4^{13}-4, \cdots, 13^{13}-13\}$. So $d|(2^{13}-2) = 2 \times 5 \times 7 \times 9 \times 13$.
$2|(n^{13}-n)$ for all n from 1 to 13.
$n^2 \equiv 1 \pmod 3 \implies n^{12} \equiv 1 \pmod 3 \implies n^{13} \equiv n \pmod 3 \implies 3|(n^{13}-n)$ for all n from 1 to 13.
$n^4 \equiv 1 \pmod 5 \implies n^{12} \equiv 1 \pmod 5 \implies n^{13} \equiv n \pmod 5 \implies 5|(n^{13}-n)$ for all n from 1 to 13.
$n^6 \equiv 1 \pmod 7 \implies n^{12} \equiv 1 \pmod 7 \implies n^{13} \equiv n \pmod 7 \implies 7|(n^{13}-n)$ for all n from 1 to 13.
$n^{12} \equiv 1 \pmod{13} \implies n^{12} \equiv 1 \pmod 1 3 \implies n^{13} \equiv n \pmod 1 3 \implies 13|(n^{13}-n)$ for all n from 1 to 13.
Note that 9 does not divide $3^{13}-3$. Hence gcd $\{2^{13}-2, 3^{13}-3, 4^{13}-4, \cdots, 13^{13}-13\}$ is $2 \times 5 \times 7 \times 3 \times 13$.

5. Let $\{a_n\}$ be a sequence of real numbers. Suppose $\{a_{sn}\}$ converges for every fixed positive integer $s > 1$.
1) If $a_{sn} \to a$ and $a_{tn} \to b$ for some fixed positive integers s and t, then is $a = b$? Justify.
2) Is the sequence $\{a_n\}$ convergent? Justify.
Solution : 1) Suppose $a_{sn} \to a$ and $a_{tn} \to b$ for some fixed positive integers s and t. Consider a subsequence $\{a_{stn}\}$. As it is a subsequence of $\{a_{sn}\}$, it converges to a. Also it is a subsequence of $\{a_{tn}\}$, therefore it converges to b. As limit is unique, $a = b$.
2) The sequence $\{a_n\}$ need not be convergent.
Define $a_n = 0$ if n is not prime and $a_n = 1$ if n is a prime. This sequence satisfies the above condition but it is not convergent.

Part I

N.B. Each question in Part I carries 2 marks.

1. If $N = 1! + 2! + 3! + \cdots + 2011!$, then the digit in the units place of the number N is
   (a) 1 (b) 3 (c) 0 (d) 9.

2. The set of all points z in the complex plane satisfying $z^2 = |z|^2$ is a
   (a) pair of points (b) circle (c) union of lines (d) line.

3. If the arithmetic mean of two numbers is 26 and their geometric mean is 10, then the equation with these two numbers as roots is
   (a) $x^2 + 52x + 100 = 0$ (b) $x^2 - 52x - 100 = 0$
   (c) $x^2 - 52x + 100 = 0$ (d) $x^2 + 52x - 10 = 0$.

4. All points lying inside the triangle with vertices at the points $(1, 3), (5, 0)$ and $(-1, 2)$ satisfy
   (a) $3x + 2y \geq 0$ (b) $2x + y - 13 \geq 0$
   (c) $2x - 3y - 12 \geq 0$ (d) $-2x + y \geq 0$.

5. For $n \geq 3$, let A be an $n \times n$ matrix. If rank of A is $n - 2$, then rank of adjoint of A is
   (a) $n - 2$ (b) 2 (c) 1 (d) 0.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an odd and differentiable function. Then for every $x_0 \in \mathbb{R}$, $f'(-x_0)$ is equal to
   (a) $f'(x_0)$ (b) $-f'(x_0)$ (c) 0 (d) None of these.

7. If $S = \{a, b, c\}$ and the relation R on the set S is given by $R = \{(a, b), (c, c)\}$, then R is
   (a) reflexive and transitive (b) reflexive but not transitive
   (c) not reflexive but transitive (d) neither reflexive nor transitive.

8. Let $a_1 = 1$, $a_{n+1} = \left(\frac{1+n}{n}\right) a_n$ for $n \geq 1$. Then the sequence $\{a_n\}$ is
   (a) divergent (b) decreasing (c) convergent (d) bounded.

9. The coefficient of x^{2n-2} in $(x-1)(x+1)(x-2)(x+2)\cdots(x-n)(x+n)$ is
   (a) 0 (b) $\frac{-n(n+1)(2n+1)}{6}$ (c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\frac{-n(n+1)}{2}$.

10. The number of roots of $5x^4 - 4x + 1 = 0$ in $[0, 1]$ is
    (a) 0 (b) 1 (c) 2 (d) 3.

Part II

N.B. Each question in Part II carries 5 marks.

1. If $n \geq 3$ is an integer and k is a real number, prove that n is equal to the sum of n^{th} powers of the roots of the equation $x^n - kx - 1 = 0$.

2. Find all positive integers n such that $(n2^n - 1)$ is divisible by 3.

3. Start with the set $S = \{3, 4, 12\}$. At any stage you may perform the following operation: Choose any two elements $a, b \in S$ and replace them by $\left(\frac{3a - 4b}{5}\right)$ and $\left(\frac{4a + 3b}{5}\right)$. Is it possible to transform the set S into the set $\{4, 6, 12\}$ by performing the above operation a finite number of times?

4. Let $a < b$. Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Let α be a real number. If $f(a) = f(b) = 0$, show that there exists $x_0 \in (a, b)$ such that $\alpha f(x_0) + f'(x_0) = 0$.

Part III

N.B. Each question in Part III carries 12 marks.

1. Let M_n be the $n \times n$ matrix with all 1's along the main diagonal, directly above the main diagonal and directly below the main diagonal and 0's everywhere else. For example,

$$M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \; M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad \text{Let } d_n = \det M_n.$$

   (a) Find d_1, d_2, d_3, d_4.
   (b) Find a formula expressing d_n in terms of d_{n-1} and d_{n-2}, for all $n \geq 3$.
   (c) Find d_{100}.

2. Let $p(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - 4x^{2n-3} + \cdots - 2nx + (2n + 1)$. Show that the polynomial $p(x)$ has no real root.

3. Let $f(x) = x^{10} + a_1x^9 + a_2x^8 + \cdots + a_{10}$ where a_i's are integers. If all the roots of $f(x)$ are from the set $\{1, 2, 3\}$, determine the number of such polynomials. Further, if $g(x)$ is the sum of all such polynomials $f(x)$, then show that the constant term of $g(x)$ is $\frac{1}{2}(3^{12} + 1) - 2^{12}$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f(x + h) - f(x) = hf'(x + \frac{1}{2}h),$$

   for all real x and h. Prove that f is a polynomial of degree atmost 2.

5. (a) Let $n = 9$. Express n as a sum of positive integers such that their product is maximum. Find the value of the maximum product.
   (b) Repeat part (a) for $n = 10$ and $n = 11$.
   (c) Given a positive integer $n \geq 6$, express n as a sum of positive integers such that their product is maximum. Find the value of the maximum product.

Solutions to Madhava Mathematics Competition 2011

Part I

N.B. Each question in Part I carries 2 marks.

1. If $N = 1! + 2! + 3! + \cdots + 2011!$, then the digit in the units place of the number N is
   (a) 1 (b) 3 (c) 0 (d) 9.
   Answer : (b)

   Note that $5! \equiv 0 \pmod{10}$. Thus, $1! + 2! + 3! + 4! = 33 \equiv 3 \pmod{10}$.

2. The set of all points z in the complex plane satisfying $z^2 = |z|^2$ is a
   (a) pair of points (b) circle (c) union of lines (d) line.
   Answer : (d)

   $z = 0$ and if $z \neq 0$ then $zz = z\bar{z}$. Hence, $z = \bar{z}$. Hence, imaginary part of $z = 0$.

3. If the arithmetic mean of two numbers is 26 and their geometric mean is 10, then the equation with these two numbers as roots is
   (a) $x^2 + 52x + 100 = 0$ (b) $x^2 - 52x - 100 = 0$
   (c) $x^2 - 52x + 100 = 0$ (d) $x^2 + 52x - 10 = 0$.
   Answer : (c)

   If the roots are α and β then $\alpha + \beta = 26$ and $\alpha\beta = 100$.

4. All points lying inside the triangle with vertices at the points $(1, 3)$, $(5, 0)$ and $(-1, 2)$ satisfy
   (a) $3x + 2y \geq 0$ (b) $2x + y - 13 \geq 0$
   (c) $2x - 3y - 12 \geq 0$ (d) $-2x + y \geq 0$.
   Answer : (a)

   Substitute the coordinates of the points.

5. For $n \geq 3$, let A be an $n \times n$ matrix. If rank of A is $n - 2$, then rank of adjoint of A is
   (a) $n - 2$ (b) 2 (c) 1 (d) 0.
   Answer : (d)

   Rank of the matrix is $n-2$. Hence, every $(n-1) \times (n-1)$ minor equals 0. Hence, every entry of the adjoint of A is 0.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an odd and differentiable function. Then for every $x_0 \in \mathbb{R}$, $f'(-x_0)$ is equal to
(a) $f'(x_0)$ (b) $-f'(x_0)$ (c) 0 (d) None of these.
Answer : (a)

Use chain rule.

7. If $S = \{a, b, c\}$ and the relation R on the set S is given by $R = \{(a, b), (c, c)\}$, then R is
(a) reflexive and transitive (b) reflexive but not transitive
(c) not reflexive but transitive (d) neither reflexive nor transitive.
Answer : (c)

$(b, b) \notin R$ Hence, R is not reflexive. However, R is transitive.

8. Let $a_1 = 1$, $a_{n+1} = \left(\frac{1+n}{n}\right) a_n$ for $n \geq 1$. Then the sequence $\{a_n\}$ is
(a) divergent (b) decreasing (c) convergent (d) bounded.
Answer : (a)

Note that $a_n = n$ for every n. Hence, $< a_n >$ is an unbounded sequence. Hence, divergent.

9. The coefficient of x^{2n-2} in

$$f(x) = (x-1)(x+1)(x-2)(x+2)\cdots(x-n)(x+n)$$

is
(a) 0 (b) $\frac{-n(n+1)(2n+1)}{6}$ (c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\frac{-n(n+1)}{2}$.
Answer : (b)

Note that $f(x) = (x^2-1)(x^2-2^2)\cdots(x^2-n^2)$. Hence the coefficient of x^{2n-2} is sum of squares of the numbers from 1 to n with negative sign.

10. The number of roots of $g(x) = 5x^4 - 4x + 1 = 0$ in $[0, 1]$ is
(a) 0 (b) 1 (c) 2 (d) 3.
Answer : (c)

$g(0) > 0$ and $g(1) > 0$ while $g(1/2) < 0$. Hence, $g(x) = 0$ has at least two roots. Note that $g'(x) < 0$ if $x^3 < 1/5$ and $g'(x) > 0$ if $x^3 > 1/5$. Hence, the function is decreasing in $(-\infty, \sqrt[3]{1/5})$ and increasing on $(\sqrt[3]{1/5}, \infty)$. At $\sqrt[3]{1/5}$ the function has absolute minimum.

Part II

N.B. Each question in Part II carries 5 marks.

1. If $n \geq 3$ is an integer and k is a real number, prove that n is equal to the sum of n^{th} powers of the roots of the equation $x^n - kx - 1 = 0$.
   Solution :
   Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the equation $x^n - kx - 1 = 0$. Therefore $\alpha_i^n = k\alpha_i + 1,\ 1 \leq i \leq n.$ [1 mark]

   Therefore $\sum_{i=1}^{n} \alpha_i^n = k \sum_{i=1}^{n} \alpha_i + n,\ 1 \leq i \leq n.$ [2 marks]

   But as $n \geq 3$, $\sum_{i=1}^{n} \alpha_i = 0$, since the coefficient of x^{n-1} is zero. Thus $\sum_{i=1}^{n} \alpha_i^n = n.$ [2 marks]

2. Find all positive integers n such that $(n2^n - 1)$ is divisible by 3.
   Solution : Note that $2^2 \equiv 1 \pmod 3$. Hence, $2^{2k} \equiv 1 \pmod 3$. Thus if n is even then $2^n \equiv 1 \pmod 3$ and if n is odd then $2^n \equiv 2 \pmod 3$. Hence, if n is even then $n2^n - 1 \equiv (n-1) \pmod 3$ and if n is odd then $n2^n - 1 \equiv 2n - 1 \pmod 3$. $3|(n2^n - 1)$ Hence $(n-1) \equiv 0 \pmod 3$ if n is even and $3|(-n-1)$ if n is odd. Hence, $n = 6k + 4$ [2 marks]
   or $n = 6k + 5$. [2 marks]

   Further, if $n = 6k + 4$ or $n = 6k + 5$ then $n|n2^n - 1$. [1 mark]

3. Start with the set $S = \{3, 4, 12\}$. At any stage you may perform the following operation: Choose any two elements $a, b \in S$ and replace them by $\left(\frac{3a - 4b}{5}\right)$ and $\left(\frac{4a + 3b}{5}\right)$. Is it possible to transform the set S into the set $\{4, 6, 12\}$ by performing the above operation a finite number of times?
   Solution :
   When we replace a and b by $a_1 = \left(\frac{3a - 4b}{5}\right)$ and $b_1 = \left(\frac{4a + 3b}{5}\right)$. The set $\{a, b, c\}$ changes to $\{a_1 = \left(\frac{3a - 4b}{5}\right), b_1 = \left(\frac{4a + 3b}{5}\right), c_1 = c\}$. The sum of squares of the elements of this set is

$\left(\frac{3a-4b}{5}\right)^2 + \left(\frac{4a+3b}{5}\right)^2 + c^2 = a^2 + b^2 + c^2$. Thus the new set $\{a_1, b_1, c_1\}$ satisfies the condition $a^2 + b^2 + c^2 = a_1^2 + b_1^2 + c_1^2$. Now the set $\{3, 4, 12\}$ has sum of squares equal to 169, where as the new set $\{4, 6, 12\}$ has sum of squares equal to 196. The two sums are different.

Hence it is not possible to transform the set $\{3, 4, 5\}$ to $\{4, 6, 12\}$. [5 marks]

Note: If the answer is no by trial and error then give 1 mark.

4. Let $a < b$. Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Let α be a real number. If $f(a) = f(b) = 0$, show that there exists $x_0 \in (a, b)$ such that $\alpha f(x_0) + f'(x_0) = 0$.
Solution :
Define $g(x) = e^{\alpha x} f(x)$. Then $g'(x) = e^{\alpha x}[\alpha f(x) + f'(x)]$. [2 marks]

As $f(a) = f(b) = 0$, we have $g(a) = g(b) = 0$. By Rolle's theorem, there exists $x_0 \in (a, b)$ such that $g'(x_0) = 0$. This implies $\alpha f(x_0) + f'(x_0) = 0$. [3 marks]

Part III

N.B. Each question in Part III carries 12 marks.

1. Let M_n be the $n \times n$ matrix with all 1's along the main diagonal, directly above the main diagonal and directly below the main diagonal and 0's everywhere else. For example,
$M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Let $d_n = \det M_n$.
(a) Find d_1, d_2, d_3, d_4. [If all are done 2 marks]
(b) Find a formula expressing d_n in terms of d_{n-1} and d_{n-2}, for all $n \geq 3$. [3 for expressing it and 3 for the proof.]
(c) Find d_{100}. [4 marks]
Solution :

- Note, $M_1 = 1$, so $\det(M_1) = d_1 = 1$.
- $M_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, so clearly $\det(M_2) = d_2 = 0$.

- Next, $M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, so clearly $\det(M_3) = 1(\det(M_2)) - 1(1) = d_3 = -1$.
- Note $M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $\det(M_4) = 1(\det(M_3)) - 1(\det(M_2)) = d_4 = -1$.

  (Some students will realize induction here and will straightaway go to general formula).
- Let $M_n = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$.

  Then, we claim: $\det(M_n) = 1(\det(M_{n-1})) - 1(\det(M_{n-2}))$ i.e.,

$$\det(M_n) = \det(M_{n-1}) - \det(M_{n-2}).$$

- The proof follows from the row-expansion formula for the determinant.

  Expanding along the first row, in M_n, we get:

$$\det(M_n) = 1(\det(M_{n-1})) - 1(\det K),$$

  where K is the following $(n-1) \times (n-1)$ matrix:

$$K = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & \cdots & 1 & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

  Again expanding along the first row, note that

$$(\det K) = 1(\det(M_{n-2})) - 1(\det K'),$$

where $K' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

Clearly, $\det K' = 0$, as all the entries in one column of K' are 0. This proves the claim.

- Now, it is easy to find that:

$$d_5 = d_4 - d_3 = (-1) - (-1) = 0,$$
$$d_6 = d_5 - d_4 = 0 - (-1) = 1,$$
$$d_7 = d_6 - d_5 = 1 - 0 = 1,$$
$$d_8 = d_7 - d_6 = 1 - 1 = 0.$$

- In fact, finding a few more terms makes the pattern obvious by looking at the following table:
  $d_1 = 1 = d_7$
  $d_2 = 0 = d_8$
  $d_3 = -1 = d_9$
  $d_4 = -1 = d_{10}$
  $d_5 = 0 = d_{11}$
  $d_6 = 1 = d_{12}$
- Thus, we can calculate d_n for any n by the following formula:

$$d_n = 1, \text{if } n \equiv 0, 1 \pmod 6,$$
$$d_n = 0, \text{if } n \equiv 2, 5 \pmod 6,$$
$$d_n = -1, \text{if } n \equiv 3, 4 \pmod 6.$$

2. Let $p(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - 4x^{2n-3} + \cdots - 2nx + (2n+1)$. Show that the polynomial $p(x)$ has no real root.
   Solution :
   If $x \leq 0$ then $p(x) > 0$. [2 marks]
   Let $x > 0$.
   $p(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - 4x^{2n-3} + \cdots - 2nx + (2n+1)$.
   $xp(x) = x^{2n+1} - 2x^{2n} + 3x^{2n-1} - 4x^{2n-2} + \cdots - 2nx^2 + (2n+1)x$.

$xp(x) + p(x) = x^{2n+1} - x^{2n} + x^{2n-1} - x^{2n-2} + \cdots + x + (2n+1)$.

$(1+x)p(x) = x\left(\dfrac{1+x^{2n+1}}{1+x}\right) + (2n+1)$.

$\Rightarrow p(x) > 0$ for $x > 0$. [10 marks]

Note: If done for an interval then maximum 2 marks.

3. Let $f(x) = x^{10} + a_1x^9 + a_2x^8 + \cdots + a_{10}$ where a_i's are integers.

If all the roots of $f(x)$ are from the set $\{1, 2, 3\}$, determine the number of such polynomials. Further, if $g(x)$ is the sum of all such polynomials $f(x)$, then show that the constant term of $g(x)$ is $\dfrac{1}{2}(3^{12}+1) - 2^{12}$.

Solution :

Note that $f(x) = (x-1)^a(x-2)^b(x-3)^c$, where $a+b+c = 10$ and $a, b, c \geq 0$ are integers. [2 marks]

If $c = 0$ then there are 11 solutions. If $c = 1$ then there are 10 solutions. If $c = 2$ then there are 9 solutions. If $c = 3$ then there are 8 solutions and so on. Thus for every c there are $11 - c$ solutions. Hence, the total number of solutions is $1 + 2 + \cdots + 11 = 66$. Hence, there are 66 such polynomials. [4 marks]

The constant term of $g(x)$ i.e a_{10} equals

$$\begin{aligned}
&\sum_{a+b+c=10} 1^a 2^b 3^c \qquad\qquad [2 \text{ marks}] \\
= \;& 3^{10} + 3^9(2+1) + 3^8(2^2 + 2(1) + 1^2) + \cdots + 3^0(2^{10} + 2^9 + \cdots + 1) \\
= \;& 3^{10}(2-1) + 3^9(2^2-1)) + 3^8(2^3-1) + \cdots + 3^0(2^{11}-1) \\
= \;& 3^{11}2(1 + \frac{2}{3} + \cdots + \left(\frac{2}{3}\right)^{10}) - \frac{1}{2}(3^{11}-1) \\
= \;& 3^{11}2(1 - (\frac{2}{3})^{11}) - \frac{1}{2}(3^{11}-1) \\
= \;& 2(3^{11} - 2^{11}) - \frac{1}{2}(3^{11}-1) \qquad\qquad [4 \text{ marks}]
\end{aligned}$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f(x+h) - f(x) = hf'(x + \frac{1}{2}h),$$

for all real x and h. Prove that f is a polynomial of degree at most 2.

Solution :

From the given condition, we have $f(x+h)-f(x-h) = 2hf'(x),\ \forall x, h$. Therefore putting $x = 0$, we get $f(h) - f(h) = 2hf'(0),\ \forall h$. Differentiating with respect to h, $f'(h) + f'(-h) = 2f'(0),\ \forall h$.

Now define $g(x) = f'(x) - f'(0)$. Then $g(0) = 0$ and $g(-x) = -g(x)$.

Again $f'(a + h) + f'(a - h) = 2f'(a)$. Putting $a + h = x, a - h = y$ in above expression, we get $f'(x) + f'(y) = 2f'(\frac{x+y}{2})$ (*)

and putting $h = a$, we get $f'(2a) + f'(0) = 2f'(a)$.

Therefore $f'(a) + f'(0) = 2f'(\frac{a}{2})$. (**)

From (*) $f'(x) + f'(y) = 2f'(\frac{x+y}{2})$

From (**) $f'(x) + f'(y) = f'(x + y) + f'(0)$.

$g(x+y) = f'(x+y) - f'(0) = f'(x) + f'(y) - 2f'(0) = [f'(x) - f'(0)] + [f'(y) - f'(0)] = g(x) + g(y)$.

Therefore $g(kx) = kg(x)$, $g(\frac{x}{n}) = \frac{1}{n}g(x)$, $g(\frac{m}{n}x) = \frac{m}{n}g(x)$.

Now g is continuous, $g(\alpha x) = \alpha g(x),\ \forall \alpha \in \mathbb{R}$. Therefore g is linear. Therefore $f'(x) - f'(0) = ax$. Therefore $f'(x) = f'(0) + ax$. Therefore $f(x) = \frac{a}{2}x^2 + f'(0)x + c$.

defining the function g is crucial hence 8 marks for that and then 4 marks for showing that it is linear.

5. (a) Let $n = 9$. Express n as a sum of positive integers such that their product is maximum. Find the value of the maximum product.
(b) Repeat part (a) for $n = 10$ and $n = 11$.
(c) Given a positive integer $n \geq 6$, express n as a sum of positive integers such that their product is maximum. Find the value of the maximum product.

Solution :

(a) 9=3+3+3.

(b) 10=2+2+3+3, 11=2+3+3+3.

Note that expressing 9, 10 and 11 carries 1 mark each.

(c) Let $n \geq 5$. Note that $n = (n - 3) + 3$ and $n < 3(n - 3)$. [6 marks]

Hence, we write n as sum of 3's till we get a number < 5. If the resulting number is 4, then we express it as $2 + 2$. If it is 2 or 3 then keep it as it is. [3 marks]

MADHAVA MATHEMATICS COMPETITION, 8 January 2012

Part I

N.B. Each question in Part I carries 2 marks.

1. Let n be a fixed positive integer. The value of k for which $\int_1^k x^{n-1}dx = \frac{1}{n}$ is
   (a) 0 (b) 2^n (c) $(\frac{2}{n})^{\frac{1}{n}}$ (d) $2^{\frac{1}{n}}$

2. Let $S = \{a, b, c\}$, $T = \{1, 2\}$. If m denotes the number of one-one functions and n denotes the number of onto functions from S to T, then the values of m and n respectively are
   (a) 6,0 (b) 0,6 (c) 5,6 (d) 0,8.

3. In the binary system, $\frac{1}{2}$ can be written as
   (a) $0.01111\cdots$ (b) $0.01\overline{000}\cdots$ (c) $0.0101\cdots$ (d) None of these

4. For the equation $|x|^2 + |x| - 6 = 0$
   (a) there is only one root. (b) the sum of the roots is -1.
   (c) the sum of the roots is 0. (d) the product of the roots is -6.

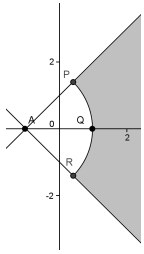
5. The value of $\lim_{n\to\infty} \frac{1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!}{(n+1)!}$
   (a) 1 (b) 2 (c) $\frac{1}{2}$ (d) does not exist.

6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f'(x) > g'(x)$ for every x. Then the graphs $y = f(x)$ and $y = g(x)$
   (a) intersect exactly once. (b) intersect at most once.
   (c) do not intersect. (d) could intersect more than once.

7. The function $|x|^3$ is
   (a) differentiable twice but not thrice at 0. (b) not differentiable at 0.
   (c) three times differentiable at 0. (d) differentiable only once at 0.

8. Let A be the $n \times n$ matrix ($n \geq 2$), whose $(i, j)^{th}$ entry is $i + j$ for all $i, j = 1, 2, \cdots, n$. The rank of A is
   (a) 2 (b) 1 (c) n (d) $n - 1$.

9. Let $A = -1 + 0i$ be the point in the complex plane. Let PQR be an arc with centre at A and radius 2. If $P = -1 + \sqrt{2} + \sqrt{2}i, Q = 1 + 0i$ and $R = -1 + \sqrt{2} - \sqrt{2}i$



   then the shaded region is given by
   (a) $|z + 1| < 2, |\arg(z + 1)| < \frac{\pi}{2}$
   (b) $|z - 1| < 2, |\arg(z - 1)| < \frac{\pi}{2}$
   (c) $|z + 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$
   (d) $|z - 1| > 2, |\arg(z + 1)| < \frac{\pi}{4}$

10. The solution of $\frac{dy}{dx} = a^{x+y}$ is
    (a) $a^x - a^{-y} = c$ (b) $a^{-x} + a^{-y} = c$ (c) $a^{-x} - a^y = c$ (d) $a^x + a^{-y} = c$

Part II

N.B. Each question in Part II carries 5 marks.

1. Let $f : [0,4] \to [3,9]$ be a continuous function. Show that there exists x_0 such that $f(x_0) = \dfrac{3x_0 + 6}{2}$.

2. Suppose a, b, c are all real numbers such that $a+b+c > 0,\ abc > 0$ and $ab+bc+ac > 0$. Show that a, b, c are all positive.

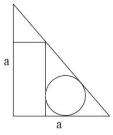
3. Let f be a continuous function on $[0,2]$ and twice differentiable on $(0,2)$. If $f(0) = 0, f(1) = 1$ and $f(2) = 2$, then show that there exists x_0 such that $f''(x_0) = 0$.

4. Integers $1, 2, \cdots, n$ are placed in such a way that each value is either bigger than all preceding values or smaller than all preceding values. In how many ways this can be done? (For example in case of $n = 5$, 3 2 4 1 5 is valid and 3 2 5 1 4 is not valid.)

Part III

N.B. Each question in Part III carries 12 marks.

1. Consider an isosceles right triangle with legs of fixed length a.



Inscribe a rectangle and a circle inside the triangle as indicated in the figure. Find the dimensions of the rectangle and the radius of the circle which make the total area of the rectangle and circle maximum.

2. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, one-one function. If there exists a positive integer n such that $f^n(x) = x$, for every $x \in \mathbb{R}$, then prove that either $f(x) = x$ or $f^2(x) = x$. (Note that $f^n(x) = f(f^{n-1}(x))$.)

3. Consider $f(x) = \dfrac{x^3}{6} + \dfrac{x^2}{2} + \dfrac{x}{3} + 1$. Prove that $f(x)$ is an integer whenever x is an integer. Determine with justification, conditions on real numbers a, b, c and d so that $ax^3 + bx^2 + cx + d$ is an integer for all integers x.

4. Suppose $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & b & d \\ 1 & b & d+1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $U = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$. Find conditions on A and U such that the system $AX = U$ has no solution.

5. Let A and B be finite subsets of the set of integers. Show that $|A + B| \geq |A| + |B| - 1$. When does equality hold?
(Here $A + B = \{x + y : x \in A,\ y \in B\}$. Also, $|S|$ denotes the number of elements in the set S.)

MADHAVA MATHEMATICS COMPETITION, 8 January 2012
Solutions and Scheme of Marking

Part I

N.B. Each question in Part I carries 2 marks.

1. Let n be a fixed positive integer. The value of k for which $\int_1^k x^{n-1}dx = \frac{1}{n}$ is
   (a) 0 (b) 2^n (c) $(\frac{2}{n})^{\frac{1}{n}}$ (d) $2^{\frac{1}{n}}$
   Solution: (d)
   $\int_1^k x^{n-1}dx = [\frac{x^n}{n}]_1^k = \frac{k^n}{n} - \frac{1}{n} = \frac{1}{n}$.
   Therefore $\frac{k^n}{n} = \frac{2}{n}$. Therefore $k^n = 2$. Hence $k = 2^{\frac{1}{n}}$.

2. Let $S = \{a, b, c\}$, $T = \{1, 2\}$. If m denotes the number of one-one functions and n denotes the number of onto functions from S to T, then the values of m and n respectively are
   (a) 6,0 (b) 0,6 (c) 5,6 (d) 0,8.
   Solution: (b)
   There can not be a one-one function from S to T because the number of elements in S is bigger than the number of elements in T. The total number of functions from S to T is 2^3. Since the constant function from S to T is not onto, the number of onto functions is $2^3 - 2 = 6$.

3. In the binary system, $\frac{1}{2}$ can be written as
   (a) $0.01111\cdots$ (b) $0.01000\cdots$ (c) $0.0101\cdots$ (d) None of these
   Solution: (a)
   Observe that $\sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}$.

4. For the equation $|x|^2 + |x| - 6 = 0$
   (a) there is only one root. (b) the sum of the roots is -1.
   (c) the sum of the roots is 0. (d) the product of the roots is -6.
   Solution: (c)
   $|x|^2 + |x| - 6 = (|x| + 3)(|x| - 2) = 0$. Therefore roots are $|x| = 2$ i.e. $x = \pm 2$. Hence the sum of the roots is 0.

5. The value of $\lim_{n\to\infty} \frac{1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!}{(n+1)!}$
   (a) 1 (b) 2 (c) $\frac{1}{2}$ (d) does not exist.
   Solution: (a)
   Observe that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (2-1) \cdot 1! + (3-1) \cdot 2! + \cdots + ((n+1)-1) \cdot n! = (2! - 1!) + (3! - 2!) + \cdots + (n+1)! - n! = (n+1)! - 1$.
   Therefore $\lim_{n\to\infty} \frac{1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!}{(n+1)!} = \lim_{n\to\infty} \frac{(n+1)! - 1}{(n+1)!} = 1$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f'(x) > g'(x)$ for every x. Then the graphs $y = f(x)$ and $y = g(x)$
   (a) intersect exactly once. (b) intersect at most once.
   (c) do not intersect. (d) could intersect more than once.

Solution: (b)

Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) > 0$. Therefore h is strictly increasing. If the graph of h intersects $X-$axis, then $f(x) = g(x)$ at exactly one point. It is possible that the graphs of f and g do not intersect. For example, $f(x) = e^x,\ g(x) = 0$. Therefore the graphs $y = f(x)$ and $y = g(x)$ intersect at most once.

7. The function $|x|^3$ is

(a) differentiable twice but not thrice at 0. (b) not differentiable at 0.

(c) three times differentiable at 0. (d) differentiable only once at 0.

Solution: (a)

Observe that $f'(0) = 0$ and $f''(0) = 0$.

$f'''(0) = \lim_{h \to 0} \dfrac{f''(h)}{h}$. Then left limit is -6 and right limit is 6. Therefore limit does not exist.

Hence f is differentiable twice but not thrice at 0.

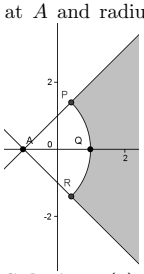
8. Let A be the $n \times n$ matrix ($n \geq 2$), whose $(i,j)^{th}$ entry is $i + j$ for all $i, j = 1, 2, \cdots, n$. The rank of A is

(a) 2 (b) 1 (c) n (d) $n - 1$.

Solution: (a)

$$\text{rank} \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & \dots & 2n \end{pmatrix} = \text{rank} \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = \text{rank} \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = 2.$$

9. Let $A = -1 + 0i$ be the point in the complex plane. Let PQR be an arc with centre at A and radius 2. If $P = -1 + \sqrt{2} + \sqrt{2}i, Q = 1 + 0i$ and $R = -1 + \sqrt{2} - \sqrt{2}i$



then the shaded region is given by

(a) $|z+1| < 2, |\arg(z+1)| < \frac{\pi}{2}$

(b) $|z-1| < 2, |\arg(z-1)| < \frac{\pi}{2}$

(c) $|z+1| > 2, |\arg(z+1)| < \frac{\pi}{4}$

(d) $|z-1| > 2, |\arg(z+1)| < \frac{\pi}{4}$

Solution: (c)

Since PQR is an arc with centre at -1 and radius 2, it satisfies the equation $|z+1| = 2$. Therefore any point in the the shaded region satisfies the condition $|z+1| > 2$.

10. The solution of $\dfrac{dy}{dx} = a^{x+y}$ is

(a) $a^x - a^{-y} = c$ (b) $a^{-x} + a^{-y} = c$ (c) $a^{-x} - a^y = c$ (d) $a^x + a^{-y} = c$

Solution: (d)

$\dfrac{dy}{dx} = a^{x+y}$. Therefore $a^{-y}dy = a^x dx$. Integrating we get, $a^x + a^{-y} = c$.

Part II

N.B. Each question in Part II carries 5 marks.

1. Let $f : [0,4] \to [3,9]$ be a continuous function. Show that there exists x_0 such that $f(x_0) = \dfrac{3x_0+6}{2}$.

   Solution:

   Define a function $g : [3,9] \to [0,4]$ as $g(x) = \dfrac{2x-6}{3}$. [2]

   Then $g \circ f : [0,4] \to [0,4]$ is a continuous function. By fixed point theorem, there exists x_0 such that $g(f(x_0)) = x_0$. i.e. $\dfrac{2f(x_0)-6}{3} = x_0$. Hence $f(x_0) = \dfrac{3x_0+6}{2}$. [3]

2. Suppose a, b, c are all real numbers such that $a+b+c > 0$, $abc > 0$ and $ab+bc+ac > 0$. Show that a, b, c are all positive.

   Solution:

   Let $a+b+c = u$,

   $ab+bc+ca = v$

   and $abc = w$

   Then a, b, c are the roots of the equation $x^3 - ux^2 + vx - w = 0$ [2]

   Now $u, v, w > 0$. If $x \leq 0$, LHS is -ve.

   $\Rightarrow$ all real roots are positive. [3]

   Hence a, b, c are all positive.

3. Let f be a continuous function on $[0,2]$ and twice differentiable on $(0,2)$. If $f(0) = 0, f(1) = 1$ and $f(2) = 2$, then show that there exists x_0 such that $f''(x_0) = 0$.

   Solution:

   Given that $f(0) = 0, f(1) = 1$ and $f(2) = 2$.

   By Mean Value Theorem, there exist $c \in (0,1)$ and $d \in (1,2)$ such that $\dfrac{f(1)-f(0)}{1-0} = 1 = f'(c)$ and $\dfrac{f(2)-f(1)}{2-1} = 1 = f'(d)$. [3]

   Therefore $f'(c) = f'(d)$.

   Applying Rolle's Theorem, there exists $x_0 \in (c,d)$, such that $f''(x_0) = 0$. [2]

4. Integers $1, 2, \cdots, n$ are placed in such a way that each value is either bigger than all preceding values or smaller than all preceding values. In how many ways this can be done? (For example in case of $n = 5$, 3 2 4 1 5 is valid and 3 2 5 1 4 is not valid.)

   Solution:

   The last term must be 1 or n and each subsequent term from the right must be either largest or smallest of those left. The first term is uniquely determined. [2]

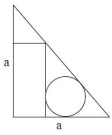
   Thus every position except 1st has two choices and so number of such sequences is 2^{n-1}.

   [3]

Part III

N.B. Each question in Part III carries 12 marks.

1. Consider an isosceles right triangle with legs of fixed length a.



Inscribe a rectangle and a circle inside the triangle as indicated in the figure. Find the dimensions of the rectangle and the radius of the circle which make the total area of the rectangle and circle maximum.

Solution: Let r be the radius of a circle and x be the smaller side of a rectangle.

We want to maximize $f(x) = (a-x)x + \pi r^2$.

Since we have the relationship

$$\frac{1}{2}(a-x)(a-x) = \frac{1}{2}r\sqrt{2}(a-x) + \frac{1}{2}r(a-x) + \frac{1}{2}r(a-x),$$

we see that, $r = \dfrac{(a-x)(a-x)}{(a-x)+(a-x)+\sqrt{2}(a-x)} = \dfrac{(a-x)}{2+\sqrt{2}}$. Then we want to maximize $f(x) = (a-x)x + \pi\dfrac{(a-x)^2}{(2+\sqrt{2})^2}$. [7]

where $0 \le x \le a$. Letting $A = \dfrac{\pi}{(2+\sqrt{2})^2}$, we can write

$f(x) = x^2(A-1) + x(a-2aA) + Aa^2$.

Differentiating f with respect to x we get, $f'(x) = 2x(A-1) + a(1-2A)$.

which has critical point at $x = \dfrac{a(1-2A)}{2(A-1)}$. Further this gives maximum since $A < 1$ gives us $f''(x) = 2(A-1) < 0$. Hence $r = \dfrac{a(4A-3)}{2(2+\sqrt{2})(A-2)}$. [5]

2. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, one-one function. If there exists a positive integer n such that $f^n(x) = x$, for every $x \in \mathbb{R}$, then prove that either $f(x) = x$ or $f^2(x) = x$. (Note that $f^n(x) = f(f^{n-1}(x))$.)

Solution: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous and one-one function. Then f is either strictly increasing or strictly decreasing. [4]

We now prove that if f is strictly increasing, then $f(x) = x$ and if f is strictly decreasing, then $f^2(x) = x$.

Case-I : Suppose f is strictly increasing and there exists x_0 such that $f(x_0) \neq x_0$.

Let $f(x_0) > x_0$. Since f is strictly increasing, $f^2(x_0) > f(x_0) > x_0$. Therefore $f^n(x_0) > x_0$, which is a contradiction.

Let $f(x_0) < x_0$. By similar arguments, we get a contradiction. [3]

Case-II : Suppose f is strictly decreasing. Then f^2 is strictly increasing. Observe that $f^n(x) = x$, for every $x \in \mathbb{R}$ implies $f^{2n}(x) = x$ for every $x \in \mathbb{R}$. Therefore $(f^2)^n(x) = x$, for every $x \in \mathbb{R}$. By applying case-I to f^2, we get $f^2(x) = x$. [5]

3. Consider $f(x) = \dfrac{x^3}{6} + \dfrac{x^2}{2} + \dfrac{x}{3} + 1$. Prove that $f(x)$ is an integer whenever x is an integer. Determine with justification, conditions on real numbers a, b, c and d so that $ax^3 + bx^2 + cx + d$ is an integer for all integers x.

Solution: Note that $f(x) = \frac{1}{6}(x^3 + 3x^2 + 2x + 6) = \frac{1}{6}(x^2 + 2)(x + 3)$. Let x be any integer. Then

$$\begin{aligned} 3|x &\Rightarrow 3|(x+3), \text{ and} \\ x = 3k \pm 1 &\Rightarrow x^2 = 9k^2 \pm 6k + 1 \Rightarrow x^2 + 2 = 9k^2 \pm 6k + 3 \Rightarrow 3|(x^2+2). \end{aligned}$$

So, in all cases, $3|(x^2 + 2)(x + 3)$. Secondly,

$$\begin{aligned} x = 2k &\Rightarrow x^2 + 2 = 4k^2 + 2 \Rightarrow 2|(x^2+2), \text{ and} \\ x = 2k - 1 &\Rightarrow x + 3 = 2k + 2 \Rightarrow 2|(x+3). \end{aligned}$$

Thus, in both cases, $2|(x^2 + 2)(x + 3)$. Therefore, since 2 and 3 are co-prime,

$6 = (2 \times 3)|(x^2 + 2)(x + 3)$. Therefore, $f(x)$ is an integer. [3]

Next, let $g(x) = ax^3 + bx^2 + cx + d$. We will show that $g(x)$ is an integer whenever x is an integer, if and only if

(i) d is an integer and there exist integers A, B, C such that
(ii) $a = A/6, \quad b = B/2, \quad c = C/6$ and such that
(iii) $3|(A + C)$ and (iv) $2|(A + B + C)$.

First, suppose that $g(x)$ is an integer whenever x is an integer. Then $g(0) = d \Rightarrow d$ is an integer. [1]

So, since $g(1) = a + b + c + d$ and $g(-1) = -a + b - c + d$, we see that $u = a + b + c$ and $v = -a + b - c$ are integers. Hence $2b = u + v = B$, say, is an integer [1]

and $2(a+c) = u - v = w$, say, is an integer. Again, $g(2) = 8a + 4b + 2c + d$ is an integer, so that $6a + 2B + w$ is an integer. [1]

Hence $6a = A$, say, is an integer. Thus $6a + 6c = 3w$ is an integer. Hence $6c = C$, say, is an integer. This proves (i) and (ii). Also, then $g(x) = \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + \frac{1}{6}Cx + d$ $= \frac{1}{6}(Ax^3 + 3Bx^2 + Cx + 6d)$. [2]

Since $w = 2(A/6 + C/6) = (A + C)/3$, is an integer, we see that $3|(A + C)$. Finally, $g(3) = \frac{1}{6}(27A + 9B + 3C + 6d) = \frac{1}{2}(9A + 3B + C + 2d)$ is an integer. Hence $2|[(A + B + C) + 8A + 2B + 2d]$ so that $2|(A + B + C)$. So, (iii) and (iv) hold. [2]

Conversely, suppose that the conditions hold for $g(x)$. Then $g(x) = \frac{1}{6}h(x)$ where $h(x) = Ax^3 + 3Bx^2 + Cx + 6d$. Let x be any integer. Then $h(x) = x(Ax^2 + 3Bx + C) + 6d$ is an integer. Clearly, $3|x \Rightarrow 3|h(x)$. Next, let $x \equiv \pm 1$ modulo 3. Then modulo 3, $x^2 \equiv 1$ so that $h(x) \equiv \pm(A + C) \equiv 0$ by (iii). So, in all cases, $3|h(x)$. Also, clearly, $2|x \Rightarrow 2|h(x)$. Next, let $x \equiv 1$ modulo 2. Then modulo 2, $x^2 \equiv 1$ so that $h(x) \equiv (A + B + C) \equiv 0$ by (iv). So, in both cases, $2|h(x)$. So, $6|h(x)$. Therefore, $g(x)$ is an integer. [4]

4. Suppose $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & b & d \\ 1 & b & d+1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $U = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$. Find conditions on A and U such that the system $AX = U$ has no solution.

Solution: We use the result that the system $AX = U$ is *inconsistent* if and only if the ranks of the matrices A and $[A|U]$ are *not* equal. We apply elementary row

transformations to the system $AX = U$ to reduce it to the form $A_1X = U_1$ where A_1 is lower triangular thus:

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & f \\ 1 & b & d & g \\ 1 & b & d+1 & h \end{array}\right]$$

$$\text{Apply } R_3 \to R_3 - R_2: \quad \left[\begin{array}{ccc|c} 2 & 1 & 0 & f \\ 1 & b & d & g \\ 0 & 0 & 1 & h-g \end{array}\right]$$

$$\text{Apply } R_2 \to R_2 - \tfrac{1}{2}R_1: \quad \left[\begin{array}{ccc|c} 2 & 1 & 0 & f \\ 0 & b-\frac{1}{2} & d & g-\frac{1}{2}f \\ 0 & 0 & 1 & h-g \end{array}\right]$$

So, if $b \neq \frac{1}{2}$, then A is invertible and the system has a unique solution for any matrix U. [6]

So, let $b = 1/2$. Apply $R_2 \to R_2 - dR_3$. Then

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & f \\ 0 & 0 & 0 & g-\frac{1}{2}f-d(h-g) \\ 0 & 0 & 1 & h-g \end{array}\right]$$

Thus rank of A_1 is 2 and rank of $[A_1|U_1]$ is 3 if and only if $g - \frac{1}{2}f - d(h-g) \neq 0$. Hence the system is not consistent if and only if $b = \frac{1}{2}$ and $g - \frac{1}{2}f - d(h-g) \neq 0$. [6]

5. Let A and B be finite subsets of the set of integers. Show that
$|A+B| \geq |A| + |B| - 1$. When does equality hold?
(Here $A+B = \{x+y : x \in A,\ y \in B\}$. Also, $|S|$ denotes the number of elements in the set S.)
Solution: We first observe that the cardinalities of A, B and $A+B$ remain invariant under translation of A or B by any integer. [3]
We now translate A so as to have $\max A = 0$ and translate B so as to have $\min B = 0$. [3]
Thus now A has all non-positive integers, B has all non-negative integers and $A \cap B = \{0\}$. Hence $|A \cup B| = |A| + |B| - 1$. As both A and B contain 0, clearly $A \cup B \subset A + B$. So $|A \cup B| \leq |A + B|$ and hence $|A| + |B| - 1 \leq |A + B|$. [2]
Claim : Equality holds i.e $A \cup B \supset A+B$ if and only if $A \cup B$ is an arithmetic progression.
(I) Let $A \cup B$ be an arithmetic progression with common difference d. For $a \in A$ and $b \in B$, there exist $k \leq 0, l \geq 0$ such that $a = kd$ and $b = ld$. Then $a + b = (k + l)d$. So $a + b \in A \cup B$ since $k \leq k + l \leq l$.
(II)Given that $A \cup B = A + B$. Let $A = \{0 > a_1 > a_2 > \cdots > a_n\}$ and $B = \{0 < b_1 < b_2 < \cdots < b_m\}$. Now observe that $|a_1| = |b_1|$ because $a_1 < a_1 + b_1 < b_1$ and $a_1 + b_1 \in A \cup B$ so that we must have $a_1 + b_1 = 0$. Now by a similar argument, $a_2 + b_1 = a_1$. It is easy to see that, $a_{j+1} + b_1 = a_j$, for all j. So A is an A. P. with common difference b_1. Similarly B is an A. P. with common difference b_1. Thus $A \cup B$ is an arithmetic progression with common difference b_1. [4]

Part I

N.B. Each question in Part I carries 2 marks.

1. If $p(x)$ is a non-constant polynomial, then $\lim_{k\to\infty} \frac{p(k+1)}{p(k)}$ is equal to
   (a) 1 (b) 0 (c) -1 (d) the leading coefficient of $p(x)$.

2. The number of continuous functions f from $[-1,1]$ to $\mathbb{R}$ satisfying $(f(x))^2 = x^2$ for all $x \in [-1,1]$ is
   (a) 2 (b) 3 (c) 4 (d) infinite.

3. Let $q \in \mathbb{N}$. The number of elements in set $\{(\cos\frac{\pi}{q} + i\sin\frac{\pi}{q})^n \mid n \in \mathbb{N}\}$ is
   (a) 1 (b) q (c) infinite (d) $2q$.

4. If $f(x) = |x|^{\frac{3}{2}}$, $\forall x \in \mathbb{R}$, then at $x = 0$,
   (a) f is not continuous (b) f is continuous but not differentiable (c) f is differentiable but f' is not continuous (d) f is differentiable and f' is continuous.

5. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are roots of the equation $x^4 + x^3 + 1 = 0$, then the value of $(1-2\alpha_1)(1-2\alpha_2)(1-2\alpha_3)(1-2\alpha_4)$ is equal to
   (a) 19 (b) 16 (c) 15 (d) 20.

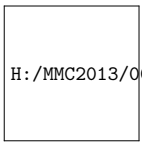
6. If A and B are 3×3 real matrices with rank $(AB) = 1$, then rank (BA) cannot be
   (a) 0 (b) 1 (c) 2 (d) 3.

7. The number of common solutions of $x^{36} - 1 = 0$ and $x^{24} - 1 = 0$ in the set of complex numbers is
   (a) 1 (b) 2 (c) 6 (d) 12.

8. If f is a one to one function from $[0,1]$ to $[0,1]$, then
   (a) f must be onto (b) f cannot be onto (c) $f([0,1])$ must contain a rational number (d) $f([0,1])$ must contain an irrational number.

9. There are 18 ways in which n identical balls can be grouped such that each group contains equal number of balls. Then the minimum value of n is
   (a) 120 (b) 180 (c) 160 (d) 90.

10. Suppose f and g are two linear functions as shown in the figure. Then $\lim_{x\to a} \frac{f(x)}{g(x)}$
    (a) is 2 (b) does not exist (c) is 3 (d) is $\frac{1}{2}$.



Part II

N.B. Each question in Part II carries 5 marks. Attempt any FOUR:

(a) Let $A = (a_{ij})$ be $n \times n$ matrix, where $a_{ij} = \max\{i, j\}$. Find the determinant of A.

(b) Assume that f is a continuous function from $[0, 2]$ to $\mathbb{R}$ and $f(0) = f(2)$. Prove that there exist x_1 and x_2 in $[0, 2]$ such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.

(c) Let $X = \{1, 2, 3, \cdots, 10\}$. Determine the number of ways of expressing X as $X = A_1 \cup A_2 \cup A_3$, where $A_1, A_2, A_3 \subseteq X$ and $A_1 \cap A_2 \cap A_3 = \phi$.

(d) For non-negative real numbers $a_1, a_2, \cdots, a_n$, show that $\frac{1}{n}\sum_{k=1}^{n} a_k e^{-a_k} \leq \frac{1}{e}$.

(e) Let A be a non-zero $1 \times n$ real matrix. Then show that the rank of $A^t A$ is 1.
[Note that A^t denotes the transpose of the matrix A.]

Part III

N.B. Each question in Part III carries 12 marks. Attempt any FIVE:

(a) Let $f(x) = \dfrac{(x-a)(x-b)}{(x-c)}$, $x \neq c$.
Find the range of f in each of the following cases:
i] $a < c < b$ ii] $c < a < b$.

(b) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.

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(c) Suppose $A_1, A_2, \cdots, A_n$ are vertices of a regular n-gon inscribed in a unit circle and P is any point on the unit circle. Prove that $\sum_{i=1}^{n} l(PA_i)^2$ is constant, where $l(PA_i)$ denotes the distance between P and A_i. [Hint: Use complex numbers.]

(d) A unit square of a chess board of size $n \times n$ gets infected if at least two of its neighbours are infected. Find the maximum number of infected unit squares if initially [i] 2 unit squares are infected, [ii] 3 unit squares are infected. Find the minimum number of unit squares that should be infected initially so that the whole chess board gets infected.
(Two unit squares are called neighbours if they share a common edge.)

(e) Let a, b, c be integers. Let $x = \dfrac{p}{q}, y = \dfrac{r}{s}$ be rational numbers satisfying $y^2 = x^3 + ax^2 + bx + c$. Show that there exists an integer t such that $q = t^2, s = t^3$.

(f) Show that the polynomial $p_n(x) = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$ has no real root if n is even and exactly one real root if n is odd.

Madhava Mathematics Competition January 6, 2013

Solutions and scheme of marking

Part I

N.B. Each question in Part I carries 2 marks.

1. If $p(x)$ is a non-constant polynomial, then $\lim_{k\to\infty} \frac{p(k+1)}{p(k)}$ is equal to
   (a) 1 (b) 0 (c) -1 (d) the leading coefficient of $p(x)$.
   Solution: (a)
   It is sufficient to find the limit in case of $p(x) = ax + b$.
   $\lim_{k\to\infty} \frac{p(k+1)}{p(k)} = 1.$

2. The number of continuous functions f from $[-1, 1]$ to $\mathbb{R}$ satisfying $(f(x))^2 = x^2$ for all $x \in [-1, 1]$ is
   (a) 2 (b) 3 (c) 4 (d) infinite.
   Solution: (c)
   The expression $(f(x))^2 = x^2$ implies $f(x) = \pm x$ or $f(x) = \pm|x|$.

3. Let $q \in \mathbb{N}$. The number of elements in set $\{(\cos\frac{\pi}{q} + i\sin\frac{\pi}{q})^n \mid n \in \mathbb{N}\}$ is
   (a) 1 (b) q (c) infinite (d) $2q$.
   Solution: (d)
   The number of elements in the given set is equal to the number of q^{th} roots of $\{\cos n\pi + i\sin n\pi \mid n \in \mathbb{N}\} = \{\pm 1\}$ which are $2q$ in number since the q^{th} roots of 1 are distinct from the q^{th} roots of -1.

4. If $f(x) = |x|^{\frac{3}{2}}$, $\forall x \in \mathbb{R}$, then at $x = 0$,
   (a) f is not continuous (b) f is continuous but not differentiable (c) f is differentiable but f' is not continuous (d) f is differentiable and f' is continuous.
   Solution: (d)
   $f'(x) = \frac{3}{2}\sqrt{x}$, if x > 0
   $= -\frac{3}{2}\sqrt{|x|}$, if x < 0
   $f'(0) = \lim_{h\to 0} \frac{|h|^{\frac{3}{2}}}{h} = 0.$
   Therefore f' exists and is continuous at 0.

5. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are roots of the equation $x^4 + x^3 + 1 = 0$, then the value of $(1 - 2\alpha_1)(1 - 2\alpha_2)(1 - 2\alpha_3)(1 - 2\alpha_4)$ is equal to
   (a) 19 (b) 16 (c) 15 (d) 20.
   Solution: (a)
   $(1 - 2\alpha_1)(1 - 2\alpha_2)(1 - 2\alpha_3)(1 - 2\alpha_4) = 1 - 2\sum \alpha_i + 4\sum \alpha_i\alpha_j - 8\sum \alpha_i\alpha_j\alpha_k + 16\alpha_1\alpha_2\alpha_3\alpha_4 = 1 - 2(-1) + 4(0) - 8(0) + 16(1) = 19.$

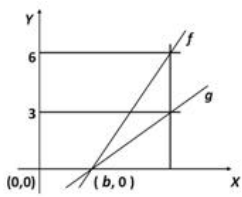
6. If A and B are 3×3 real matrices with rank $(AB) = 1$, then rank (BA) cannot be
   (a) 0 (b) 1 (c) 2 (d) 3.
   Solution: (d)
   If the rank of BA is 3, then the determinant of BA is nonzero. Then the determinant of AB is nonzero and hence the matrix AB is also invertible, which is not possible.

7. The number of common solutions of $x^{36} - 1 = 0$ and $x^{24} - 1 = 0$ in the set of complex numbers is
(a) 1 (b) 2 (c) 6 (d) 12.
Solution: (d)
The number of common solutions of $x^{36} - 1 = 0$ and $x^{24} - 1 = 0$ in the set of complex numbers is the $\gcd(36, 24) = 12$.

8. If f is a one to one function from $[0, 1]$ to $[0, 1]$, then
(a) f must be onto (b) f cannot be onto (c) $f([0, 1])$ must contain a rational number (d) $f([0, 1])$ must contain an irrational number.
Solution: (d)
Because f is one to one, $f([0, 1])$ and $[0, 1]$ are equivalent sets. Also $f([0, 1]) \subseteq [0, 1]$. Now $[0, 1]$ is uncountable and the set of rational numbers is countable. Hence $f([0, 1])$ must contain an irrational number.

9. There are 18 ways in which n identical balls can be grouped such that each group contains equal number of balls. Then the minimum value of n is
(a) 120 (b) 180 (c) 160 (d) 90.
Solution: (b)
The total number of required ways= the total number of factors of n.
$180 = 2^2 \times 3^2 \times 5$. Therefore the total number of factors of 180 is $3 \times 3 \times 2 = 18$.

10. Suppose f and g are two linear functions as shown in the figure. Then $\lim_{x \to b} \frac{f(x)}{g(x)}$
(a) is 2 (b) does not exist (c) is 3 (d) is $\frac{1}{2}$.



Solution: (a)
By L′ Hospital Rule,

$$\lim_{x \to b} \frac{f(x)}{g(x)} = \lim_{x \to b} \frac{f'(x)}{g'(x)} = \frac{6}{3} = 2.$$

Part II

N.B. Each question in Part II carries 5 marks. Attempt any FOUR:

(a) Let $A = (a_{ij})$ be $n \times n$ matrix, where $a_{ij} = \max\{i, j\}$. Find the determinant of A.
Solution: Let $a_{ij} = \max\{i, j\}$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 2 & 3 & 4 & \dots & n \\ 3 & 3 & 3 & 4 & \dots & n \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ n & n & n & n & \dots & n \end{pmatrix}$$

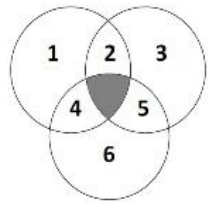
Performing the operations $R_n - R_{n-1},\ R_{n-1} - R_{n-2}, \cdots, R_2 - R_1$ on A we get

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix}. \qquad [1]$$

Hence the determinant of A is $(-1)^{n+1}n$.

(b) Assume that f is a continuous function from $[0,2]$ to $\mathbb{R}$ and $f(0) = f(2)$. Prove that there exist x_1 and x_2 in $[0,2]$ such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.
Solution: Define $g(x) = f(x+1) - f(x)$ for $x \in [0,1]$.
Note that $g(1) = f(2) - f(1)$ and $g(0) = f(1) - f(0) = -g(1)$.
Hence by Intermediate Value Property, there exists $x_0 \in [0,1]$ such that $g(x_0) = 0$.
Therefore $f(x_0 + 1) = f(x_0)$.

(c) Let $X = \{1, 2, 3, \cdots, 10\}$. Determine the number of ways of expressing X as $X = A_1 \cup A_2 \cup A_3$, where $A_1, A_2, A_3 \subseteq X$ and $A_1 \cap A_2 \cap A_3 = \phi$.
Solution: Each number from the set X has 6 choices as shown in the figure.



Therefore the total number of required ways is 6^{10}.

(d) For non-negative real numbers $a_1, a_2, \cdots, a_n$, show that $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} a_k e^{-a_k} \le \frac{1}{e}$.

Solution: Let $f(x) = xe^{-x}$ for $x \ge 0$.
Then $f'(x) = e^{-x} - xe^{-x} = 0$ implies $e^{-x}(1-x) = 0$. Therefore $x = 1$ is the only critical point. Note that $f''(1) = -\dfrac{1}{e} < 0$.
Hence $f(1) = \dfrac{1}{e}$ is the maximum value of f.
Therefore $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} a_k e^{-a_k} \le \frac{1}{n}\ n\ \frac{1}{e} = \frac{1}{e}$.

(e) Let A be a non-zero $1 \times n$ real matrix. Then show that the rank of A^tA is 1.
[Note that A^t denotes the transpose of the matrix A.]
Solution: Let $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$. Then

$$A^tA = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} = \begin{pmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2^2 & \cdots & a_2a_n \\ \vdots & \ddots & \ddots & \vdots \\ a_na_1 & a_na_2 & \cdots & a_n^2 \end{pmatrix}.$$

Without loss of generality we can assume that $a_1 \neq 0$. Performing row operations $R_2 \to R_2 - \dfrac{a_2}{a_1}R_1, R_3 \to R_3 - \dfrac{a_3}{a_1}R_1, \cdots, R_n \to R_n - \frac{a_n}{a_1}R_1$, we obtain

$$A^tA = \begin{pmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence the rank of A^tA is 1.

Part III

N.B. Each question in Part III carries 12 marks. Attempt any FIVE:

(a) Let $f(x) = \dfrac{(x-a)(x-b)}{(x-c)}, \quad x \neq c.$

Find the range of f in each of the following cases:

i] $a < c < b$ ii] $c < a < b$.

Solution: Let $f(x) = \dfrac{(x-a)(x-b)}{(x-c)} = y.$

Therefore $(x-a)(x-b) = y(x-c)$.

Therefore $x^2 - (a+b+y)x + (ab+yc) = 0$.

This quadratic equation has solution $x = \dfrac{(a+b+y) \pm \sqrt{(a+b+y)^2 - 4(ab+yc)}}{2}$.

This x is real if and only if the term in the square root is non-negative.

Therefore we need $(a+b+y)^2 - 4(ab+yc) \geq 0$.

Therefore $a^2 + b^2 + y^2 + 2ab + 2by + 2ay - 4ab - 4yc \geq 0$.

Therefore $y^2 + 2(a+b-2c)y + (a+b-2c)^2 + (a-b)^2 - (a+b-2c)^2 \geq 0$.

Therefore $[y + (a+b-2c)]^2 \geq (a+b-2c)^2 - (a-b)^2$.

Simplifying this we get, $[y + (a+b-2c)]^2 \geq 4(c-b)(c-a)$.

i] If $a < c < b$, then the product $(c-b)(c-a)$ is negative and the above inequality is true for all real values of y. Hence the range of f is $\mathbb{R}$.

ii] If $c < a < b$, then the product $(c-b)(c-a)$ is positive.

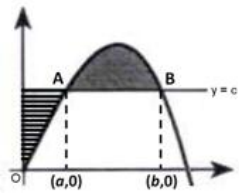
We have $y+(a+b-2c) \geq 2\sqrt{(c-b)(c-a)}$ or $y+(a+b-2c) \leq -2\sqrt{(c-b)(c-a)}$.

Therefore $y \geq (2c-a-b)+2\sqrt{(c-b)(c-a)}$ or $y \leq (2c-a-b)-2\sqrt{(c-b)(c-a)}$.

Hence the range of f is

$(-\infty, (2c-a-b) - 2\sqrt{(c-b)(c-a)}] \cup [(2c-a-b) + 2\sqrt{(c-b)(c-a)}, \infty)$

(b) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



Solution: Suppose $A(a,c)$ and $B(b,c)$ are the points on the curve as shown in the figure. Therefore they satisfy $c = 2a - 3a^3 = 2b - 3b^3$.

The areas of the two shaded regions are equal if

$$ca - \int_0^a (2x - 3x^3)\, dx = \int_a^b (2x - 3x^3)\, dx - c(b-a).$$

Therefore $ca - [x^2 - 3\frac{x^4}{4}]_0^a = [x^2 - 3\frac{x^4}{4}]_a^b - cb + ca$

Therefore $ca - a^2 + 3\frac{a^4}{4} = b^2 - 3\frac{b^4}{4} - a^2 + 3\frac{a^4}{4} - cb + ca.$

Therefore $b^2 - 3\frac{b^4}{4} - bc = 0.$

Substituting the value of c we get, $b^2 - 3\frac{b^4}{4} - b(2b - 3b^3) = 0.$

Therefore $b^2 - 3\frac{b^4}{4} - 2b^2 + 3b^4 = 0.$

Therefore $\frac{9}{4}b^4 = b^2.$

Hence cancelling b^2 from both sides we get, $b^2 = \frac{4}{9}$.

Therefore $b = \frac{2}{3}$.

Substituting the value of b in the expression $c = 2b - 3b^3$ we get, $c = \frac{4}{9}$.

(c) Suppose $A_1, A_2, \cdots, A_n$ are vertices of a regular n-gon inscribed in a unit circle and P is any point on the unit circle. Prove that $\sum_{i=1}^{n} l(PA_i)^2$ is constant, where $l(PA_i)$ denotes the distance between P and A_i. [Hint: Use complex numbers.]

Solution: Here $n \geq 3$. Let $z_1, z_2, \cdots, z_n$ be the roots of unity.

Therefore $\sum_{k=1}^{n} z_k = 0.$

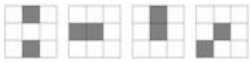
$\sum_{i=1}^{n} l(PA_i)^2 = \sum_{k=1}^{n} |z - z_k|^2 = \sum_{k=1}^{n} (z - z_k)\overline{(z - z_k)} = \sum_{k=1}^{n} (z - z_k)(\overline{z} - \overline{z_k}) =$

$\sum_{k=1}^{n} |z|^2 - \sum_{k=1}^{n} z\overline{z_k} - \sum_{k=1}^{n} z_k\overline{z} + \sum_{k=1}^{n} |z_k|^2 = n + n = 2n.$

(d) A unit square of a chess board of size $n \times n$ gets infected if at least two of its neighbours are infected. Find the maximum number of infected unit squares if initially [i] 2 unit squares are infected, [ii] 3 unit squares are infected. Find the minimum number of unit squares that should be infected initially so that the whole chess board gets infected.

(Two unit squares are called neighbours if they share a common edge.)

Solution: If initially 2 unit squares are infected, then the maximum number of



infected unit squares is 4.

If initially 3 unit squares are infected, then the maximum number of infected unit squares is 9.

Observe that n diagonal infected squares can infect the whole chess board. Hence n are enough to infect the entire board.

To show that n are needed we observe that total perimeter of the infected region can not increase as the virus spreads over the board. When a new square is infected at least two of its boundary edges are absorbed into the infected region and at the most two boundary edges are added to it. Therefore if there are less than n squares that are initially infected the total perimeter of the infected region will be less than $4n$ initially and it will remain less than $4n$ as the virus spreads. Hence

the entire board with the perimeter $4n$ will never be entirely infected.

(e) Let a, b, c be integers. Let $x = \frac{p}{q}, y = \frac{r}{s}$ be rational numbers satisfying $y^2 = x^3 + ax^2 + bx + c$. Show that there exists an integer t such that $q = t^2, s = t^3$.

Solution: Let $x = \frac{p}{q}, y = \frac{r}{s}$ be the rational solutions of $y^2 = x^3 + ax^2 + bx + c$ where $(p, q) = 1 = (r, s), q > 0, s > 0$.

Substituting the values of x and y, we get

$$\frac{r^2}{s^2} = \frac{p^3}{q^3} + \frac{ap^2}{q^2} + \frac{bp}{q} + c$$

This implies $r^2q^3 = p^3s^2 + ap^2qs^2 + bpq^2s^2 + cq^3s^2$ (*)

This implies $q|p^3s^2$. Therefore $q|s^2$. (1)

So, $q^2|ap^2qs^2$ implies $q^2|p^3s^2$ by (*)

Hence $q^2|s^2$. Therefore $q|s$. (2)

So, $q^3|ap^2qs^2$ and $q^3|bpq^2s^2$.

By (*) $q^3|p^3s^2$. Therefore $q^3|s^2$. (3)

Also (*) implies $s^2|r^2q^3$. Therefore $s^2|q^3$. (4)

By (3) and (4), $s^2 = q^3$.

But by (2), $q|s$. Therefore $s = qt$ for some $t \in \mathbb{N}$.

So, $s^2 = q^3$ implies $q^2t^2 = q^3$. Therefore $q = t^2$ and $s = qt = t^3$.

(f) Show that the polynomial $p_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ has no real root if n is even and exactly one real root if n is odd.

Solution: Observe that $p_n'(x) = p_{n-1}(x)$.

Note that $p_1(x) = 1 + x$ has exactly one real root $x = -1$.

Consider $p_2(x) = 1 + x + \frac{x^2}{2!}$. The quadratic equation $p_2(x) = 0$ i.e. $x^2 + 2x + 2 = 0$ has no real root.

Let $p_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. This is polynomial of odd degree. Therefore it has at least one real root. If $p_3(x)$ has two real roots a and b, then between a and b, there is a root of $p_3'(x) = p_2(x)$. But $p_2(x)$ has no real root. Hence $p_3(x)$ has exactly one real root.

Let $p_3(a) = 0$. Note that a is a negative real number. Observe that a is a critical point of $p_4(x)$. Now $p_4''(x) = p_3'(x) = p_2(x)$. Then $p_2(a) = p_3(a) - \frac{a^3}{3!} = -\frac{a^3}{3!} > 0$. Therefore $p_4''(a) > 0$. Hence $p_4(x)$ has minimum value at a.

Therefore $p_4(a) \leq p_4(x)$ for all x.

But $p_4(a) = p_3(a) + \frac{a^4}{4!} > 0$. Therefore $p_4(x) > 0$ for all x. Hence $p_4(x)$ has no real root. Repeating the similar argument we can prove the result.

MADHAVA MATHEMATICS COMPETITION

Date: January 5, 2014 Time: 12 noon- 3 p.m.

Part I

N.B. Each question in Part I carries 2 marks. [20]

1. If $x^3 - x + 1 = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3$, then (a_0, a_1, a_2, a_3) equals
   A. $(1, -1, 0, 1)$ **B.** $(7, 6, 10, 1)$ **C.** $(7, 11, 12, 6)$ **D.** $(7, 11, 6, 1)$

2. Suppose $f(x)$ and $g(x)$ are real-valued differentiable functions such that $f'(x) \geq g'(x)$ for all x in $[0, 1]$. Which of the following is necessarily true?
   A. $f(1) \geq g(1)$
   B. $f - g$ has no maximum on $[0, 1]$
   C. $f(1) - g(1) \geq f(0) - g(0)$
   D. $f + g$ is a non-decreasing function on $[0, 1]$

3. The equation $x^4 + x^2 - 1 = 0$ has
   A. two positive and two negative roots
   B. all positive roots
   C. one positive, one negative and two non-real roots
   D. no real root

4. Let n be a natural number. Let A and B be $n \times n$ matrices. If A is invertible, then which of the following is necessarily true?
   A. $rank(AB) < rank(B)$
   B. $rank(AB) > rank(B)$
   C. $rank(AB) = rank(B)$
   D. $rank(AB) < rank(A)$

5. Let X be a set and A, B, C be its subsets. Which of the following is necessarily true?
   A. $A - (A - B) = B$
   B. $A - (B \cap C) = (A - B) \cap (A - C)$
   C. $A - (B \cup C) = (A - B) \cup (A - C)$
   D. $B - (A - B) = B$

6. For a real number x we let $[x]$ denote the largest integer not exceeding x. For a natural number n, let $a_n = \frac{[n\sqrt{2}]}{n}$. The limit $\lim_{n\to\infty} a_n$
   A. equals 0 **B.** equals $[\sqrt{2}]$ **C.** equals $\sqrt{2}$ **D.** does not exist

7. Let M be a two-digit natural number. Let N be the natural number whose digits are that of M but are in the reverse order. Which of the following CANNOT be the sum of M and N?
   A. 181 **B.** 165 **C.** 121 **D.** 154

8. The value of $\lim_{x\to 1} \frac{\int_1^x e^{t^2} dt}{x^2 - 1}$ is
   A. 0 **B.** 1 **C.** e **D.** $e/2$

9. Let n be any positive integer and $1 \leq x_1 < x_2 < \cdots < x_{n+1} \leq 2n$, where each x_i is an integer. Which of the following must be true?
   (I) There is an i such that x_i is a square of an integer.
   (II) There is an i such that $x_{i+1} = x_i + 1$.
   (III) There is an i such that x_i is prime.
   A. I only **B.** II only **C.** I and II only **D.** II and III only

10. Two real numbers x and y are chosen uniformly at random from the interval $[0, 1]$. Find the probability that $2x > y$.
    A. 1/4 **B.** 1/2 **C.** 2/3 **D.** 3/4

Part II

N.B. Each question in Part II carries 6 marks. [30]

1. Let A be an 8×3 matrix in which every entry is either 1 or -1, and no two rows are identical. Find the rank of A.

2. Find all pairs (x, y) of integers such that $y^2 = x(x+1)(x+2)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is a decreasing function. If a, b, c are real numbers with $a < c < b$, prove that $(b-c)f(a) + (c-a)f(b) \leq (b-a)f(c)$.

4. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be a polynomial with integer coefficients such that a_0, a_3 and $f(1)$ are odd. Show that f has no rational root.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x-y) = f(x)f(y)$ and $f(x) \neq 0$ for all x. Find $f(3)$.

Part III

1. Prove that the equation $e^x - ln(x) - 2^{2014} = 0$ has exactly two positive real roots. [12]

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a non-constant function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that

   (a) $f(x) \neq 0$ for all $x \in \mathbb{R}$;

   (b) $f(x) > 0$ for all $x \in \mathbb{R}$;

   (c) if f is differentiable at 0, then f is differentiable on $\mathbb{R}$ and there exists some real number β such that $f(x) = \beta^x$ for all $x \in \mathbb{R}$. [12]

3. Let n be a natural number. Suppose $P_1, P_2, \cdots, P_n$ are points on a circle of radius 1. Prove that

$$\sum_{1 \leq i < j \leq n} d(P_i, P_j)^2 \leq n^2 \,,$$

   where for points X and Y in the plane, we denote by $d(X, Y)$ the distance between them. Prove that equality can hold for every natural number n. [13]

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(0) = 0$. Suppose that $|f(z) - f(w)| = |z - w|$ for any $w \in \{0, 1, i\}$ and $z \in \mathbb{C}$. Prove that $f(z) = \alpha z$ or $f(z) = \alpha \overline{z}$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. [13]

MADHAVA MATHEMATICS COMPETITION, JANUARY 5, 2014

N.B. Each question in Part I carries 2 marks. [20]

1. If $x^3 - x + 1 = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3$, then (a_0, a_1, a_2, a_3) equals

   A. $(1, -1, 0, 1)$ **B.** $(7, 6, 10, 1)$ **C.** $(7, 11, 12, 6)$ **D.** $(7, 11, 6, 1)$

   Solution: (D)

   Using Taylor series expansion for $f(x) = x^3 - x + 1$ about $x = 2$, we get $a_r = \dfrac{f^{(r)}(2)}{r!}$. Then $a_0 = 7, a_1 = 11, a_2 = 6, a_3 = 1$.

2. Suppose $f(x)$ and $g(x)$ are real-valued differentiable functions such that $f'(x) \geq g'(x)$ for all x in $[0, 1]$. Which of the following is necessarily true?

   A. $f(1) \geq g(1)$ **C.** $f(1) - g(1) \geq f(0) - g(0)$

   B. $f - g$ has no maximum on $[0, 1]$ **D.** $f + g$ is a non-decreasing function on $[0, 1]$

   Solution: (C)

   The condition $f'(x) - g'(x) \geq 0$ implies that $f(x) - g(x)$ is an increasing function on $[0, 1]$. Therefore $f(1) - g(1) \geq f(0) - g(0)$.

3. The equation $x^4 + x^2 - 1 = 0$ has

   A. two positive and two negative roots **C.** one positive, one negative and two non-real roots

   B. all positive roots **D.** no real root

   Solution: (C)

   Put $x^2 = y$. Then the equation becomes $y^2 + y - 1 = 0$. Solving this equation, we get $x^2 = y = \dfrac{-1 \pm \sqrt{5}}{2}$. When $x^2 = \dfrac{-1 + \sqrt{5}}{2}$, we get two real values of x, one positive and other negative. But no real x exists such that $x^2 = \dfrac{-1 - \sqrt{5}}{2}$. Hence The equation $x^4 + x^2 - 1 = 0$ has one positive, one negative and two non-real roots.

4. Let n be a natural number. Let A and B be $n \times n$ matrices. If A is invertible, then which of the following is necessarily true?

   A. $rank(AB) < rank(B)$ **C.** $rank(AB) = rank(B)$

   B. $rank(AB) > rank(B)$ **D.** $rank(AB) < rank(A)$

   Solution: (C)

   Since A is invertible, it is product of elementary matrices. The matrix AB can be obtained from B by performing elementary row transformations. Therefore $rank(AB) = rank(B)$.

5. Let X be a set and A, B, C be its subsets. Which of the following is necessarily true?

   A. $A - (A - B) = B$ **C.** $A - (B \cup C) = (A - B) \cup (A - C)$

   B. $A - (B \cap C) = (A - B) \cap (A - C)$ **D.** $B - (A - B) = B$

   Solution: (D)

   Note that $A - B = A \cap B'$. Also, $(A - B)' = (A \cap B')' = A' \cup B$.

   Hence $B - (A - B) = B \cap (A - B)' = B \cap (A' \cup B) = (B \cap A') \cup B = B$.

6. For a real number x we let $[x]$ denote the largest integer not exceeding x. For a natural number n, let $a_n = \dfrac{\lceil n\sqrt{2} \rceil}{n}$. The limit $\lim\limits_{n \to \infty} a_n$

   A. equals 0 **B.** equals $\lceil \sqrt{2} \rceil$ **C.** equals $\sqrt{2}$ **D.** does not exist

   Solution: (C)

   Note that $(n\sqrt{2}) - 1 \leq \lceil n\sqrt{2} \rceil \leq n\sqrt{2}$.

   Therefore $\sqrt{2} - \dfrac{1}{n} \leq \dfrac{\lceil n\sqrt{2} \rceil}{n} \leq \sqrt{2}$. Taking limit as $n \to \infty$, we get The limit $\lim\limits_{n \to \infty} a_n = \sqrt{2}$.

7. Let M be a two-digit natural number. Let N be the natural number whose digits are that of M but are in the reverse order. Which of the following CANNOT be the sum of M and N?

   A. 181 **B.** 165 **C.** 121 **D.** 154

   Solution: (A)

   If $M = 10a_1+a_0$, then $N = 10a_0+a_1$. Therefore $M+N = 10(a_0+a_1)+(a_0+a_1) = (11)(a_0+a_1)$. Hence $M+N$ is divisible by 11, but 181 is not divisible by 11.

8. The value of $\lim\limits_{x\to 1} \dfrac{\int_1^x e^{t^2}dt}{x^2-1}$ is

   A. 0 **B.** 1 **C.** e **D.** $e/2$

   Solution: (D)

   This limit is in the $\dfrac{0}{0}$ form. Then by L'Hospital rule,

   $\lim\limits_{x\to 1} \dfrac{\int_1^x e^{t^2}dt}{x^2-1} = \lim\limits_{x\to 1} \dfrac{\frac{d}{dx}\int_1^x e^{t^2}dt}{2x} = \lim\limits_{x\to 1} \dfrac{e^{x^2}}{2x} = \dfrac{e}{2}.$

9. Let n be any positive integer and $1 \le x_1 < x_2 < \cdots < x_{n+1} \le 2n$, where each x_i is an integer. Which of the following must be true?

   (I) There is an i such that x_i is a square of an integer.
   (II) There is an i such that $x_{i+1} = x_i + 1$.
   (III) There is an i such that x_i is prime.

   A. I only **B.** II only **C.** I and II only **D.** II and III only

   Solution: (B)

   Let $S_{2n} = \{1, 2, \ldots, 2n\}$. If we choose the 6 integers $2, 3, 5, 6, 7, 8$ from S_{10}, then none of them is a square. So (I) is false for $n = 5$. If we choose the 6 integers $1, 4, 6, 8, 9, 10$ from S_{10}, then none of them is a prime. So (III) is false for $n = 5$.

   Now (II) is clearly true for $n = 2$. Also, for $n \ge 2$, suppose, if possible, $n+1$ integers x_i can be chosen such that

   $$1 \le x_1 < x_2 < \cdots < x_{n+1} \le 2n,$$

   and such that no two of them differ by 1. Then $x_2 \ge x_1 + 2$, $x_3 \ge x_2 + 2$, ... $x_{n+1} \ge x_n + 2$. Adding these, we get $x_{n+1} \ge x_1 + 2n \ge 2n + 1 > 2n$. This contradicts the choice of x_{n+1}. So (II) is true for every n.

10. Two real numbers x and y are chosen uniformly at random from the interval $[0, 1]$. Find the probability that $2x > y$.

    A. 1/4 **B.** 1/2 **C.** 2/3 **D.** 3/4

    Solution: (D)

    The probability that $2x > y$ is the area of the region in the unit square below the line $2x = y$. Thus the required probability is 3/4.

Part II

N.B. Each question in Part II carries 6 marks. [30]

1. Let A be an 8×3 matrix in which every entry is either 1 or -1, and no two rows are identical. Find the rank of A.

   Solution: The given conditions imply that the rows of A must be the following triplets, in some order.

   $$(1,1,1), \quad (1,1,-1), \quad (1,-1,1), \quad (-1,1,1),$$
   $$(1,-1,-1), \quad (-1,1,-1), \quad (-1,-1,1), \quad (-1,-1,-1).$$

To see this, let (a, b, c) be a triplet satisfying the given conditions. Then each of a, b, c can be chosen in one two ways (1 or -1). So there are exactly $2^3 = 8$ such distinct triplets : they are listed above. So they are the rows of A, in some order, as A has size 8×3. [3]
Also, the 3×3 submatrix of A having the triplets $(1, 1, 1)$, $(-1, 1, 1)$, $(-1, -1, 1)$ as rows in some order is non-singular. So the rank of A is 3. [3]

2. Find all pairs (x, y) of integers such that $y^2 = x(x + 1)(x + 2)$.

Solution: If $x < -2$, then there is no solution. [1]
If $x = 0, -1, -2$, then $y = 0$. [1]
If $x \geq 1$, then $\gcd(x + 1, x(x + 2)) = 1$. Therefore $(x + 1)$ and $x(x + 2)$ are both perfect squares. But, $x(x + 2) = x^2 + 2x = (x + 1)^2 - 1$. This implies $(x + 1)^2$ and $(x + 1)^2 - 1$ are consecutive numbers which are squares. This is not possible. [4]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is a decreasing function. If a, b, c are real numbers with $a < c < b$, prove that $(b - c)f(a) + (c - a)f(b) \leq (b - a)f(c)$.

Solution: Consider any $c \in (a, b)$.
Applying Lagrange's Mean Value Theorem to f on $[a, c]$ we get, there exists $x_1 \in (a, c)$ such that $\dfrac{f(c) - f(a)}{c - a} = f'(x_1)$. [2]
Applying Lagrange's Mean Value Theorem to f on $[c, b]$ we get, there exists $x_2 \in (c, b)$ such that $\dfrac{f(b) - f(c)}{b - c} = f'(x_2)$. [2]
Now f' is a decreasing function. Therefore $x_1 < x_2$ implies $f'(x_2) < f'(x_1)$.
Hence $\dfrac{f(b) - f(c)}{b - c} < \dfrac{f(c) - f(a)}{c - a}$. Therefore $f(b) - f(c)c - a < f(c) - f(a)b - c$.
This implies $(b - c)f(a) + (c - a)f(b) \leq (b - c + c - a)f(c) = (b - a)f(c)$. [2]

4. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be a polynomial with integer coefficients such that a_0, a_3 and $f(1)$ are odd. Show that f has no rational root.

Solution: Suppose f has a rational root, say, $\dfrac{p}{q}$. Then $f(\dfrac{p}{q}) = 0$. Therefore $a_0q^3 + a_1pq^2 + a_2p^2q + a_3p^3 = 0$.
This implies $q|a_3$ and $p|a_0$. Since a_0, a_3 are odd, p, q are also odd. [3]
Also $a_0q^3 + a_1pq^2 + a_2p^2q + a_3p^3 = 0$ implies $a_0q^3 + a_1pq^2 + a_2p^2q + a_3p^3 \equiv 0 (mod 2)$.
Since p, q are odd, we have $a_0 + a_1 + a_2 + a_3 \equiv 0 (mod\ 2)$. This is contradiction because $f(1) = a_0 + a_1 + a_2 + a_3$ is odd. Hence f has no rational root. [3]

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x - y) = f(x)f(y)$ and $f(x) \neq 0$ for all x. Find $f(3)$.

Solution I: If $y = x$, then $f(0) = [f(x)]^2$. If, in particular, $x = 0$, then $f(0) = [f(0)]^2$. But, $f(x) \neq 0$. Therefore $f(0) = 1$. Now $[f(x)]^2 = 1$ implies $f(x) = \pm 1$. [2]
Put $x = 1, y = \dfrac{1}{2}$, Then $f(\dfrac{1}{2}) = f(1)\dfrac{1}{2}$. But, $f(x) \neq 0$. Therefore $f(1) = 1$. [2]
Also $f(2 - 1) = f(2)f(1) = f(1)$. Therefore $f(2) = 1$.
Also $f(3 - 2) = f(3)f(2) = f(1)$. Therefore $f(3) = 1$. [2]
OR
Solution II: Put $x = 3, y = 1.5$, then $f(1.5) = f(3 - 1.5) = f(3)f(1.5)$. Hence $f(3) = 1$. [6]

Part III

1. Prove that the equation $e^x - ln(x) - 2^{2014} = 0$ has exactly two positive real roots. [12]

Solution: Let $f(x) = e^x - \log x - 2^{2014},\ x > 0$. So $f'(x) = e^x - \frac{1}{x}$, and $f''(x) = e^x + \frac{1}{x^2}$. [2]
So $f'(1) = e - 1 > 0$. As $\lim_{x\to 0+} f'(x) = -\infty$, by the continuity of f', $f'(x) = 0$ for some value $x = a$. [2]
Since $f''(x) > 0$ for $x > 0$, f' strictly increases for $x > 0$. Therefore f has a unique critical point $x = a$. Note that f has minimum value at $x = a$ as $f''(a) > 0$. This minimum value is negative because, $f(a) < f(1) < 0$. [2]
As $\lim_{x\to 0+} \log x = -\infty$, it can be seen that $f(x)$ is positive when x is near 0. Also, as $\lim_{x\to\infty} e^{-x} \log x = 0$, it can be seen that $f(x)$ is positive when x is large. Thus there exist b, c with $0 < b < a$ and $a < c$ such that $f(b) > 0$, $f(a) < 0$ and $f(c) > 0$. Hence by continuity of f, $f(x) = 0$ has exactly two real roots : one root in each of the intervals (b, a) and (a, c). [6]
Note: Students may draw graphs for the proof. If only graph of $f(x)$ is drawn, give 4 marks. Further if there is more explanation with the graph, additional 4 marks may be given. If the argument is complete, all marks may be given.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a non-constant function satisfying $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that

   (a) $f(x) \neq 0$ for all $x \in \mathbb{R}$;

   (b) $f(x) > 0$ for all $x \in \mathbb{R}$;

   (c) If f is differentiable at 0, then f is differentiable on $\mathbb{R}$ and there exists some real number β such that $f(x) = \beta^x$ for all $x \in \mathbb{R}$. [12]

Solution: (a) Suppose for some real number x_0, we have $f(x_0) = 0$. Then for any $x \in \mathbb{R}$, $f(x) = f(x - x_0 + x_0) = f(x - x_0)f(x_0) = 0$. Therefore f is a constant function, which is a contradiction. Hence $f(x) \neq 0$ for all $x \in \mathbb{R}$ [2]

(b) Observe that $f(x) = f(\frac{x}{2})f(\frac{x}{2}) = [f(\frac{x}{2})]^2$.
Therefore $f(x) > 0$ for all $x \in \mathbb{R}$. [2]

(c) Suppose $f'(0)$ exists. Also given condition implies $f(0) = 1$.
Then for any $x \in \mathbb{R}$,
$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$
$= \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h}$
$= \lim_{h\to 0} f(x)[\frac{f(h) - 1}{h}]$
$= \lim_{h\to 0} f(x)[\frac{f(h) - f(0)}{h}] = f(x)f'(0)$.
Therefore f is differentiable on $\mathbb{R}$ and $f'(x) = f(x)f'(0)$. [4]
This implies $\frac{f'}{f} = k = f'(0)$.
Integrating both sides with respect to x, we get,
$\log f(x) = kx + c$.
Now, $x = 0$ implies $\log f(0) = \log 1 = c = 0$.
Hence, $f(x) = e^{kx} = \beta^x$, where $\beta = e^k$. [4]

3. Let n be a natural number. Suppose $P_1, P_2, \cdots, P_n$ are points on a circle of radius 1. Prove that

$$\sum_{1 \le i < j \le n} d(P_i, P_j)^2 \le n^2 \,,$$

where for points X and Y in the plane, we denote by $d(X, Y)$ the distance between them. Prove that equality can hold for every natural number n. [13]

Solution: Consider a circle of radius 1 with center at origin. If $\overline{r_1}, \overline{r_2}, \cdots, \overline{r_n}$ are position vectors of $P_1, P_2, \cdots, P_n$ respectively, then we want to prove that $\sum(\overline{r_i} - \overline{r_j}) \cdot (\overline{r_i} - \overline{r_j}) \le 2n^2$. [3]

Now $\sum(\overline{r_i} - \overline{r_j}) \cdot (\overline{r_i} - \overline{r_j})$
$= (\overline{r_1} - \overline{r_2}) \cdot (\overline{r_1} - \overline{r_2}) + (\overline{r_1} - \overline{r_3}) \cdot (\overline{r_1} - \overline{r_3}) + \cdots + (\overline{r_1} - \overline{r_n}) \cdot (\overline{r_1} - \overline{r_n}) + (\overline{r_2} - \overline{r_1}) \cdot (\overline{r_2} - \overline{r_1}) +$
$(\overline{r_2} - \overline{r_3}) \cdot (\overline{r_2} - \overline{r_3}) + \cdots + (\overline{r_2} - \overline{r_n}) \cdot (\overline{r_2} - \overline{r_n}) + \cdots + (\overline{r_n} - \overline{r_{n-1}}) \cdot (\overline{r_n} - \overline{r_{n-1}})$
$= 2(n-1)(\overline{r_1} \cdot \overline{r_1} + \overline{r_2} \cdot \overline{r_2} + \cdots + \overline{r_n} \cdot \overline{r_n}) - 2\sum_{i \ne j} 2\overline{r_i} \cdot \overline{r_j}$
$= 2(n-1)n - 2\sum_{i \ne j} 2\overline{r_i} \cdot \overline{r_j}$ [4]
$= 2(n-1)n + 2(\overline{r_1} \cdot \overline{r_1} + \overline{r_2} \cdot \overline{r_2} + \cdots + \overline{r_n} \cdot \overline{r_n}) - 2(\overline{r_1} \cdot \overline{r_1} + \overline{r_2} \cdot \overline{r_2} + \cdots + \overline{r_n} \cdot \overline{r_n}) - 2\sum_{i \ne j} 2\overline{r_i} \cdot \overline{r_j}$
$= 2n^2 - 2n + 2n - (\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n}) \cdot (\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n})$
$= 2n^2 - (\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n}) \cdot (\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n}) \le 2n^2$. [4]

Equality holds if $\overline{r_1} + \overline{r_2} + \cdots + \overline{r_n} = \overline{0}$. [2]

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(0) = 0$. Suppose that $|f(z) - f(w)| = |z - w|$ for any $w \in \{0, 1, i\}$ and $z \in \mathbb{C}$. Prove that $f(z) = \alpha z$ or $f(z) = \alpha \overline{z}$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. [13]

Solution: Let $\alpha = f(1)$ and $\beta = f(i)$.
By the hypothesis, $|f(z)| = |z|, |f(z) - \alpha| = |z - 1|$ and $|f(z) - \beta| = |z - i|$ for all $z \in \mathbb{C}$.
In particular, by substituting $z = 1, i$ in the above equalities, we obtain
$|\alpha| = 1 = |\beta|,\ |\alpha - \beta| = \sqrt{2}$. [4]
We can write
$\alpha^2 + \beta^2$
$= \alpha^2|\beta|^2 + \beta^2|\alpha|^2$
$= \alpha^2\beta\overline{\beta} + \beta^2\alpha\overline{\alpha}$
$= \alpha\beta(\alpha\overline{\beta} + \beta\overline{\alpha})$
$= \alpha\beta(\alpha\overline{\alpha} + \beta\overline{\beta} - (\alpha - \beta)(\overline{\alpha} - \overline{\beta}))$
$= \alpha\beta(|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2)$
$= \alpha\beta(1 + 1 - 2) = 0,$
yielding $\beta = \epsilon\alpha$, where $\epsilon = \pm i$. [4]
Simplifying $|f(z) - \alpha|^2 = |z - 1|^2$, we get $\overline{\alpha} f(z) + \alpha\overline{f(z)} = z + \overline{z}$ and
simplifying $|f(z) - \beta|^2 = |z - i|^2$, we get $\overline{\alpha} f(z) - \alpha\overline{f(z)} = -\epsilon i z + \epsilon i \overline{z}$ for all $z \in \mathbb{C}$.
Adding up these equalities, we obtain

$$2\overline{\alpha} f(z) = (1 - \epsilon i)z + (1 + \epsilon i)\overline{z}.$$

If $\epsilon = i$, then $f(z) = \alpha z$ and if $\epsilon = -i$, then $f(z) = \alpha \overline{z}$ for all $z \in \mathbb{C}$. [5]

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S.P.College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 13/12/2015 Max. Marks:100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let $A(t)$ denote the area bounded by the curve $y = e^{-|x|}$, the $X-$ axis and the straight lines $x = -t, x = t$, then $\lim_{t\to\infty} A(t)$ is
   A) 2 B) 1 C) 1/2 D) e.

2. How many triples of real numbers (x, y, z) are common solutions of the equations $x + y = 2, \quad xy - z^2 = 1$?
   A) 0 B) 1 C) 2 D) infinitely many.

3. For non-negative integers x, y the function $f(x, y)$ satisfies the relations $f(x, 0) = x$ and $f(x, y + 1) = f(f(x, y), y)$. Then which of the following is the largest?
   A) $f(10, 15)$ B) $f(12, 13)$ C) $f(13, 12)$ D) $f(14, 11)$.

4. Suppose p, q, r, s are $1, 2, 3, 4$ in some order. Let $x = \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s}}}}$
   We choose p, q, r, s so that x is as large as possible, then s is
   A) 1 B) 2 C) 3 D) 4.

5. Let $f(x) = \begin{cases} 3x + x^2 & \text{if } x < 0 \\ x^3 + x^2 & \text{if } x \geq 0. \end{cases}$ Then $f''(0)$ is
   A) 0 B) 2 C) 3 D) None of these.

6. There are 8 teams in pro-kabaddi league. Each team plays against every other exactly once. Suppose every game results in a win, that is, there is no draw. Let $w_1, w_2, \cdots, w_8$ be number of wins and $l_1, l_2, \cdots, l_8$ be number of loses by teams $T_1, T_2, \cdots, T_8$, then
   A) $w_1^2 + \cdots + w_8^2 = 49 + (l_1^2 + \cdots + l_8^2)$. B) $w_1^2 + \cdots + w_8^2 = l_1^2 + \cdots + l_8^2$.
   C) $w_1^2 + \cdots + w_8^2 = 49 - (l_1^2 + \cdots + l_8^2)$. D) None of these.

7. The remainder when $m + n$ is divided by 12 is 8, and the remainder when $m - n$ is divided by 12 is 6. If $m > n$, then the remainder when mn divided by 6 is
   A) 1 B) 2 C) 3 D) 4 .

8. Let $A = \begin{pmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \ddots & \vdots & \\ (n-1)n+1 & (n-1)n+2 & \dots & n^2 \end{pmatrix}$. Select any entry and call it x_1. Delete row and column containing x_1 to get an $(n-1) \times (n-1)$ matrix. Then select any entry from the remaining entries and call it x_2. Delete row and column containing x_2 to get $(n-2) \times (n-2)$ matrix. Perform n such steps. Then $x_1 + x_2 + \cdots + x_n$ is
   A) n B) $\frac{n(n+1)}{2}$ C) $\frac{n(n^2+1)}{2}$ D) None of these.

9. The maximum of the areas of the rectangles inscribed in the region bounded by the curve $y = 3 - x^2$ and $X-$axis is
A) 4 B) 1 C) 3 D) 2.

10. How many factors of $2^5 3^6 5^2$ are perfect squares?
A) 24 B) 20 C) 30 D) 36.

Part II

N.B. Each question in Part II carries 6 marks.

1. How many 15−digit palindromes are there in each of which the product of the non-zero digits is 36 and the sum of the digits is equal to 15? (A string of digits is called a palindrome if it reads the same forwards and backwards. For example 04340, 6411146.)

2. Let H be a finite set of distinct positive integers none of which has a prime factor greater than 3. Show that the sum of the reciprocals of the elements of H is smaller than 3. Find two different such sets with sum of the reciprocals equal to 2.5.

3. Let A be an $n \times n$ matrix with real entries such that each row sum is equal to one. Find the sum of all entries of A^{2015}.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0,\ f'(x) > f(x)$ for all $x \in \mathbb{R}$. Prove that $f(x) > 0$ for all $x > 0$.

5. Give an example of a function which is continuous on $[0, 1]$, differentiable on $(0, 1)$ and not differentiable at the end points. Justify.

Part III

1. There are some marbles in a bowl. A, B and C take turns removing one or two marbles from the bowl, with A going first, then B, then C, then A again and so on. The player who takes the last marble from the bowl is the loser and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can B and C work together and force A to lose? [12]

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f'(0)$ exists. Suppose $\alpha_n \neq \beta_n, \forall n \in \mathbb{N}$ and both sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge to zero. Define $D_n = \dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$.
Prove that $\lim\limits_{n\to\infty} D_n = f'(0)$ under EACH of the following conditions:

   (a) $\alpha_n < 0 < \beta_n,\ \forall n \in \mathbb{N}$.

   (b) $0 < \alpha_n < \beta_n$ and $\dfrac{\beta_n}{\beta_n - \alpha_n} \leq M,\ \ \forall n \in \mathbb{N}$, for some $M > 0$.

   (c) $f'(x)$ exists and is continuous for all $x \in (-1, 1)$.

   [13]

3. Let $f(x) = x^5$. For $x_1 > 0$, let $P_1 = (x_1, f(x_1))$. Draw a tangent at the point P_1 and let it meet the graph again at point P_2. Then draw a tangent at P_2 and so on. Show that the ratio $\dfrac{A(\triangle P_n P_{n+1} P_{n+2})}{A(\triangle P_{n+1} P_{n+2} P_{n+3})}$ is constant. [12]

4. Let $p(x)$ be a polynomial with positive integer coefficients. You can ask the question: What is $p(n)$ for any positive integer n? What is the minimum number of questions to be asked to determine $p(x)$ completely? Justify. [13]

MADHAVA MATHEMATICS COMPETITION, December 2015
Solutions and Scheme of Marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let $A(t)$ denote the area bounded by the curve $y = e^{-|x|}$, the $X-$ axis and the straight lines $x = -t, x = t$, then $\lim_{t\to\infty} A(t)$ is
A) 2 B) 1 C) 1/2 D) e.
Solution: (A)
As $f(x) = e^{-|x|}$ is an even function, $A(t) = 2\int_{-t}^{0}\int_{0}^{e^x} 1\ dydx = 2\int_{-t}^{0} e^x dx = 2(e^0 - e^{-t})$ $\to 2$ as $t \to \infty$. **OR** $A(t) = 2\int_0^t e^x\ dx = -2(e^{-t} - 1) \to 2$ as $t \to \infty$.

2. How many triples of real numbers (x, y, z) are common solutions of the equations
$x + y = 2, \quad xy - z^2 = 1$?
A) 0 B) 1 C) 2 D) infinitely many.
Solution: (B)
$xy = 1+z^2 \geq 1$ so that $-4xy \leq -4$. Hence $(x-y)^2 = (x+y)^2-4xy = 4-4xy \leq 4-4 = 0$. So $x = y$. Thus the only solution is $x = 1, y = 1, z = 0$.

3. For non-negative integers x, y the function $f(x, y)$ satisfies the relations $f(x, 0) = x$ and $f(x, y + 1) = f(f(x, y), y)$. Then which of the following is the largest?
A) $f(10, 15)$ B) $f(12, 13)$ C) $f(13, 12)$ D) $f(14, 11)$.
Solution: (D)
$f(x, 1) = f(f(x, 0), 0) = f(x, 0) = x$. Inductively $f(x, y) = x$ for all integers $y \geq 0$.

4. Suppose p, q, r, s are $1, 2, 3, 4$ in some order. Let $x = \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s}}}}$.
We choose p, q, r, s so that x is as large as possible, then s is
A) 1 B) 2 C) 3 D) 4.
Solution: (C)
For x to be the largest, p, q, r, s should be $\min\{1, 2, 3, 4\}$, $\max\{1, 2, 3, 4\}$, $\min\{2, 3\}$, $\max\{2, 3\}$ respectively. So $s = 3$.

5. Let $f(x) = \begin{cases} 3x + x^2 & \text{if } x < 0 \\ x^3 + x^2 & \text{if } x \geq 0. \end{cases}$ Then $f''(0)$ is
A) 0 B) 2 C) 3 D) None of these.
Solution: (D)
$f'_-(x) = 3 + 2x$, for $x < 0$ and $f'_+(x) = 3x^2 + 2x$, for $x \geq 0$. So $f'(0)$ does not exist because $f'_-(0) = 3 \neq 0 = f'_+(0)$.

6. There are 8 teams in pro-kabaddi league. Each team plays against every other exactly once. Suppose every game results in a win, that is, there is no draw. Let $w_1, w_2, \cdots, w_8$ be number of wins and $l_1, l_2, \cdots, l_8$ be number of loses by teams $T_1, T_2, \cdots, T_8$, then
A) $w_1^2 + \cdots + w_8^2 = 49 + (l_1^2 + \cdots + l_8^2)$. B) $w_1^2 + \cdots + w_8^2 = l_1^2 + \cdots + l_8^2$.
C) $w_1^2 + \cdots + w_8^2 = 49 - (l_1^2 + \cdots + l_8^2)$. D) None of these.
Solution: (B)
Note that $w_i + l_i = 7$ for all i and $\sum w_i - \sum l_i = \sum(w_i - l_i) = 0$. Then $\sum w_i^2 - \sum l_i^2 = \sum(w_i + l_i)(w_i + l_i) = 7\sum(w_i - l_i) = 0$.

7. The remainder when $m+n$ is divided by 12 is 8, and the remainder when $m-n$ is divided by 12 is 6. If $m > n$, then the remainder when mn divided by 6 is
A) 1 B) 2 C) 3 D) 4 .
Solution: (A)
Note that $m + n \equiv 8(\text{mod } 12)$ and $m - n \equiv 6(\text{mod } 12)$. Adding these congruences, we get, $2m \equiv 2(\text{mod } 12)$. This implies $m \equiv 1(\text{mod } 6)$. Similarly by subtracting, we get, $n \equiv 1($ mod 6). Thus $mn \equiv 1(\text{mod } 6)$.

8. Let $A = \begin{pmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \ddots & \vdots & \\ (n-1)n+1 & (n-1)n+2 & \dots & n^2 \end{pmatrix}$. Select any entry and call it x_1. Delete row and column containing x_1 to get an $(n-1) \times (n-1)$ matrix. Then select any entry from the remaining entries and call it x_2. Delete row and column containing x_2 to get $(n-2) \times (n-2)$ matrix. Perform n such steps. Then $x_1 + x_2 + \cdots + x_n$ is
A) n B) $\frac{n(n+1)}{2}$ C) $\frac{n(n^2+1)}{2}$ D) None of these.
Solution: (C)
For $n = 4$, $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1+4 & 2+4 & 3+4 & 4+4 \\ 1+8 & 2+8 & 3+8 & 4+8 \\ 1+12 & 2+12 & 3+12 & 4+12 \end{pmatrix}$. Note that $S = x_1 + x_2 + x_3 + x_4 =$ $a + (b + 4) + (c + 8) + (d + 12)$ where a, b, c, d is a permutation of $1, 2, 3, 4$. So $S = a+b+c+d+24 = 10+24 = 34$. Thus S is the same for all stated choices of $x_1, \dots, x_n$. Hence taking x_i's as the main diagonal elements,

$$\begin{aligned} S &= 1 + (n+2) + (2n+3) + \ldots + [n(n-1)+n] \\ &= [n + 2n + \ldots + n(n-1)] + [1 + 2 + \ldots + n] \\ &= \frac{n(n-1)n}{2} + \frac{n(n+1)}{2} = \frac{n(n^2+1)}{2}. \end{aligned}$$

9. The maximum of the areas of the rectangles inscribed in the region bounded by the curve $y = 3 - x^2$ and $X-$axis is
A) 4 B) 1 C) 3 D) 2.
Solution: (A)
By symmetry, let the base of the rectangle be segment with ends $-x, x$ and height y. Then area $A(x) = 2xy = 2x(3 - x^2) = 6x - 2x^3$ and $A'(x) = 6 - 6x^2$, $A'(x) = -12x$. So $A'(x) = 0$ at $x^2 = 1$ i.e. $x = 1$; and $A''(1) = -12 < 0$. So $A(x)$ is maximum at $x = 1$ with maximum value $A(1) = 4$.

10. How many factors of $2^5 3^6 5^2$ are perfect squares?
A) 24 B) 20 C) 30 D) 36.
Solution: (A)
Factors that are perfect squares will be of the form $d = 2^a 3^b 5^c$ where $a = 0, 2$ or 4, $b = 0, 2, 4$ or 6, and $c = 0$ or 2. Thus there are $3 \times 4 \times 2$ possible divisors that are perfect squares.

Part II

N.B. Each question in Part II carries 6 marks.

1. How many 15−digit palindromes are there in each of which the product of the non-zero digits is 36 and the sum of the digits is equal to 15? (A string of digits is called a palindrome if it reads the same forwards and backwards. For example 04340, 6411146.)
Solution: The first 7 digits completely determine the number. Since the sum of digits is 15, the 8^{th} digit is odd and is a factor of 36. Note that the product of all non-zero digits (except the digit in the 8^{th} place) is a square using the definition of palindrome. Hence

the 8^{th} digit cannot be 3 because the product in that case is 12. So the 8^{th} digit is either 1 or 9. **[2]**

Case 1: If the 8^{th} digit is 1 then the digits in first seven places can either be a permutation of 1,1,2,3,0,0,0 or 1,6,0,0,0,0,0 because these are the only possibilities with sum 7 and product 6.

Number of permutations of 1,1,2,3,0,0,0 is $\frac{7!}{2!3!}$.

Number of permutations of 1,6,0,0,0,0,0 is $\frac{7!}{5!}$. **[2]**

Case 2: If the 8^{th} digit is 9 then the digits in first seven places will be a permutation of 1,2,0,0,0,0,0 because this is the only possibility with sum 3 and product 2..

Number of permutations of 1,2,0,0,0,0,0 is $\frac{7!}{5!}$.

Thus the number of required 15 digit palindromes with product of nonzero digits 36 and sum of digits 15 is $\frac{7!}{2!3!}+\frac{7!}{5!}+\frac{7!}{5!}$. **[2]**

2. Let H be a finite set of distinct positive integers none of which has a prime factor greater than 3. Show that the sum of the reciprocals of the elements of H is smaller than 3. Find two different such sets with sum of the reciprocals equal to 2.5.

Solution: The given condition implies that every $n \in H,\ n$ is of the form $n = 2^{\alpha}3^{\beta}$, $\alpha, \beta \geq 0$. Since H is finite, $\exists k \in \mathbb{N}$ such that $\alpha \leq k,\ \beta \leq k$ for each $n \in H$. This implies

$$\sum_{n\in H}\frac{1}{n} \leq 1+\sum_{i=1}^{k}\frac{1}{2^i}+\sum_{j=1}^{k}\frac{1}{3^j}+\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{1}{2^i3^j}$$

$$= 1+\sum_{i=1}^{k}\frac{1}{2^i}+\sum_{j=1}^{k}\frac{1}{3^j}+\left(\sum_{i=1}^{k}\frac{1}{2^i}\right)\left(\sum_{j=1}^{k}\frac{1}{3^j}\right)$$

$$= \left(1+\frac{1}{2}+\cdots+\frac{1}{2^k}\right)\left(1+\frac{1}{3}+\cdots+\frac{1}{3^k}\right) \quad \textbf{[3]}$$

$$= (\frac{1-\frac{1}{2^{k+1}}}{1-1/2})(\frac{1-\frac{1}{3^{k+1}}}{1-1/3}) < \ (\frac{1}{1/2})(\frac{1}{2/3}) = 2(\frac{3}{2}) = 3. \quad \textbf{[1]}$$

Let $H = \{1, 2, 3, 4, 6, 8, 12, 24\}$. Then $\sum_{n\in H}\frac{1}{n} = 2.5$. **[1]**

Let $H = \{1, 2, 3, 4, 6, 8, 12, 36, 72\}$. Then $\sum_{n\in H}\frac{1}{n} = 2.5$. **[1]**

Any other correct choices for H, also carries one mark each.

3. Let A be an $n \times n$ matrix with real entries such that each row sum is equal to one. Find the sum of all entries of A^{2015}.

Solution: Let A be an $n \times n$ matrix with real entries such that each row sum is equal to one. This implies

$$A\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}. \quad \textbf{[3]}$$

By repeated use of this, we get

$$A^{2015}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}. \quad \textbf{[2]}$$

So each row sum of A^{2015} is equal to one. Hence the sum of all entries of A^{2015} is n. **[1]**

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0,\ f'(x) > f(x)$ for all $x \in \mathbb{R}$.

Prove that $f(x) > 0$ for all $x > 0$.

Solution: By data, $f'(0) > f(0) = 0$. So $\lim_{x\to 0+} \frac{f(x)}{x} > 0$. Hence $\exists\ \delta > 0$, such that $f(x) > 0$ for all $x \in (0, \delta)$. **[1]**

Now if there exists $x_0 > 0$ such that $f(x_0) \leq 0$, then by intermediate value property, there exists $x_1 \geq \delta$ such that $f(x_1) = 0$. Let $c = \inf A$, $A = \{x \mid x > 0\ ,\ f(x) = 0\}$. **[1]**

Clearly, as f is continuous and c is $\inf A$, $f(x) \geq 0$, for $x \in [0, c]$. So $c \geq \delta$. By the property of infimum, there exists a sequence $\{x_n\}$ of points which converges to c, and $x_n > 0$ and $f(x_n) = 0$. By continuity, $f(c) = \lim_{n\to\infty} f(x_n) = 0$. **[1]**

This implies that in $(0, c)$, $f(x) > 0$ and $f(0) = f(c) = 0$. Hence by Rolle's theorem, $f'(b) = 0$ for some b, $0 < b < c$. **[2]**

But then $f(b) < f'(b) = 0$ which is a contradiction. Hence $f(x) > 0$ for all $x > 0$. **[1]**

Second method. As before, $\exists\ \delta > 0$, such that $f(x) > 0$ for all $x \in (0, \delta)$. **[1]**

Let $S = \{x \mid f(t) > 0$ for $t \in (0, x).\}$ Then $\delta \in S$ so that S is non-empty. Let $m = \sup S$. If $m = \infty$, we are done. Let, if possible, $m < \infty$. Now $f(x) > 0$, $x \in (0, m)$. By continuity, $f(m) = \lim_{t\to m-} f(t) \geq 0$. **[2]**

So for all $x \in [0, m]$, $f'(x) > f(x) \geq 0$ so that $f'(x) > 0$. Hence f is strictly increasing on $[0, m]$, in particular, $f(m) > f(m/2) > 0$. Since $f(m) > 0$, by continuity, there exists $\delta_1 > 0$ such that $f(x) > 0$ in $[m, m + \delta_1)$. So, $f(x) > 0$, for $x \in (0, m + \delta_1)$. Thus $m + \delta_1 \in S$, which is a contradiction since $m = \sup S$. Hence $m = \infty$. **[3]**

5. Give an example of a function which is continuous on $[0, 1]$, differentiable on $(0, 1)$ and not differentiable at the end points. Justify.

Solution: $f(x) = \sqrt{x - x^2}$ for $x \in [0, 1]$. **[3]**

Then $f'(x)$ exists on $(0, 1)$, $f'(x) = \dfrac{1 - 2x}{2\sqrt{x - x^2}}$. **[1]**

But $f'(0) = f'(1) = \infty$. **[2]**

Note: Any other correct example with justification will carry full marks.

Part III

1. There are some marbles in a bowl. A, B and C take turns removing one or two marbles from the bowl, with A going first, then B, then C, then A again and so on. The player who takes the last marble from the bowl is the loser and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can B and C work together and force A to lose? [12]

Solution: We claim that B and C can force A to lose for all N except
N = 2; 3; 4; 7; or 8.
At N = 2, A leaves 1.
At N = 3 or 4, A leaves 2.
At N = 7 or 8, A leaves 6 after which B and C must leave 2, 3 or 4.
For N = 5 or 6, regardless of what A takes, B and C can work it so that when A's turn arrives there is only one marble left.
For N = 9 or 10, A must leave 7, 8 or 9 from which B and C can force 5 or 6. **[6]**
For N = $4k$ where $k > 2$, A must leave either $4k - 1$ or $4k - 2$ from which B and C can force $4(k - 1) + 1$ or $4(k - 2) + 2$.
For N = $4k+1$, A must leave either $4k$ or $4k-1$ from which B and C can force $4(k-1)+2$ or $4(k - 1) + 1$.
For N = $4k+2$, A must leave either $4k+1$ or $4k$ from which B and C can force $4(k-1)+2$ or $4(k - 1) + 1$.
For N = $4k + 3$, A must leave either $4k + 2$ or $4k + 1$ from which B and C can force $4(k - 1) + 2$.

In all cases for N $\geq$ 11, A will always be faced with a new value of the form $4t+1$ or $4t+2$ on his next turn eventually forcing him to N = 5 or 6 and a loss. **[6]**

2. Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f'(0)$ exists. Suppose $\alpha_n \neq \beta_n, \forall n\in\mathbb{N}$ and both sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge to zero. Define $D_n = \dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}$.
Prove that $\lim\limits_{n\to\infty} D_n = f'(0)$ under EACH of the following conditions:

   (a) $\alpha_n < 0 < \beta_n,\ \forall n\in\mathbb{N}$.

   (b) $0<\alpha_n<\beta_n$ and $\dfrac{\beta_n}{\beta_n-\alpha_n}\leq M,\ \ \forall n\in\mathbb{N}$, for some $M>0$.

   (c) $f'(x)$ exists and is continuous for all $x\in(-1,1)$. [13]

Solution: Let $\epsilon>0$ be given. Given that $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}$ exists.

(a) Given that $\alpha_n<0<\beta_n,\ \forall n\in\mathbb{N}$. Since $\alpha_n\to 0$ and $\beta_n\to 0$, we have

$f'(0)=\lim\limits_{n\to\infty}\dfrac{f(\alpha_n)-f(0)}{\alpha_n}$ and $f'(0)=\lim\limits_{n\to\infty}\dfrac{f(\beta_n)-f(0)}{\beta_n}$.

There exist $n_1,n_2\in\mathbb{N}$ such that $|f(\alpha_n)-f(0)-\alpha_n f'(0)|<|\alpha_n|\epsilon=-\alpha_n\epsilon,\ \ \forall n\geq n_1$ and

$|f(\beta_n)-f(0)-\beta_n f'(0)|<\ |\beta_n|\epsilon=\beta_n\epsilon,\ \ \forall n\geq n_2$.

Let $n_0=\max\{n_1,n_2\}$. Then $\forall n\geq n_0$, we get

$|f(\beta_n)-f(\alpha_n)-(\beta_n-\alpha_n)f'(0)|$

$\leq |f(\beta_n)-f(0)-\beta_n f'(0)|+|f(\alpha_n)-f(0)-\alpha_n f'(0)|<\ (\beta_n-\alpha_n)\epsilon$.

Thus $|\dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(0)|<\ \epsilon,\ \ \forall n\geq n_0$.

Hence $\lim\limits_{n\to\infty} D_n = f'(0)$. **[4]**

(b) Given that $0<\alpha_n<\beta_n$ and $\dfrac{\beta_n}{\beta_n-\alpha_n}\leq M,\ \ \forall n\in\mathbb{N}$, for some $M>0$.
Since $\alpha_n<\beta_n$, observe that $\dfrac{\alpha_n}{\beta_n-\alpha_n}\leq M,\ \ \forall n\in\mathbb{N}$.
Similar to part (a), there exist $n_1,n_2\in\mathbb{N}$ such that $|f(\alpha_n)-f(0)-\alpha_n f'(0)|<\ |\alpha_n|\epsilon=\alpha_n\epsilon,\ \ \forall n\geq n_1$ and $|f(\beta_n)-f(0)-\beta_n f'(0)|<\ |\beta_n|\epsilon=\beta_n\epsilon,\ \ \forall n\geq n_2$.
Let $n_0=\max\{n_1,n_2\}$. Then $\forall n\geq n_0$, we get

$|\dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(0)|$

$=|\dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-(\dfrac{\beta_n-\alpha_n}{\beta_n-\alpha_n})f'(0)|$

$=|\dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-\dfrac{\beta_n f'(0)}{\beta_n-\alpha_n}+\dfrac{\alpha_n f'(0)}{\beta_n-\alpha_n}|$

$=|(\dfrac{f(\beta_n)-f(0)-\beta_n f'(0)}{\beta_n-\alpha_n})-(\dfrac{f(\alpha_n)-f(0)-\alpha_n f'(0)}{\beta_n-\alpha_n})|$

$\leq|\dfrac{f(\beta_n)-f(0)-\beta_n f'(0)}{\beta_n-\alpha_n}|+|\dfrac{f(\alpha_n)-f(0)-\alpha_n f'(0)}{\beta_n-\alpha_n}|$

$<\ (\dfrac{\beta_n}{\beta_n-\alpha_n})\epsilon+(\dfrac{\alpha_n}{\beta_n-\alpha_n})\epsilon\leq 2M\epsilon$.

Hence $\lim\limits_{n\to\infty} D_n = f'(0)$. **[5]**

(c) Given that $f'(x)$ exists and is continuous for all $x\in(-1,1)$.
By Lagrange's Mean Value Theorem, for every positive integer n, there exists c_n between

α_n and β_n such that
$D_n = \dfrac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = f'(c_n)$.
Since $\alpha_n \to 0$ and $\beta_n \to 0$, $c_n \to 0$. It is given that $f'(x)$ is continuous.
Therefore $\lim\limits_{n\to\infty} f'(c_n) = f'(0)$.
Hence $\lim\limits_{n\to\infty} D_n = f'(0)$. **[4]**

Second method for (a), (b). Let $g(x) = f(x) - f(0) - xf'(0)$ on $\mathbb{R}$. Then $g(0) = 0$ and $g'(0) = \lim\limits_{x\to 0} \dfrac{g(x)}{x} = \lim\limits_{x\to 0}\left[\dfrac{f(x) - f(0)}{x} - f'(0)\right] = 0$. Let $E_n = \dfrac{g(\beta_n) - g(\alpha_n)}{\beta_n - \alpha_n}$. Then $E_n = D_n - f'(0)$.

(a) Let $\alpha_n < 0 < \beta_n$. Then $0 < -\alpha_n < \beta_n - \alpha_n$. Hence $0 < \dfrac{-\alpha_n}{\beta_n - \alpha_n} < 1$. Also, here $\beta_n - \alpha_n > \beta_n$ so that $0 < \dfrac{\beta_n}{\beta_n - \alpha_n} < 1$. Hence

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \frac{g(\beta_n)}{\beta_n}\left(\frac{\beta_n}{\beta_n - \alpha_n}\right) + \lim_{n\to\infty} \frac{g(\alpha_n)}{\alpha_n}\left(\frac{-\alpha_n}{\beta_n - \alpha_n}\right) = 0,$$

as the sequences in brackets are both bounded and $g'(0) = 0$. **[4]**

(b) Let $0 < \alpha_n < \beta_n$ and $0 < \dfrac{\beta_n}{\beta_n - \alpha_n} < M$. Then $\beta_n - \alpha_n > 0$ and so

$0 < \dfrac{\alpha_n}{\beta_n - \alpha_n} < \dfrac{\beta_n}{\beta_n - \alpha_n} < M$. Hence

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \frac{g(\beta_n)}{\beta_n}\left(\frac{\beta_n}{\beta_n - \alpha_n}\right) - \lim_{n\to\infty} \frac{g(\alpha_n)}{\alpha_n}\left(\frac{\alpha_n}{\beta_n - \alpha_n}\right) = 0, \text{ as in (a).}$$ **[5]**

3. Let $f(x) = x^5$. For $x_1 > 0$, let $P_1 = (x_1, f(x_1))$. Draw a tangent at the point P_1 and let it meet the graph again at point P_2. Then draw a tangent at P_2 and so on. Show that the ratio $\dfrac{A(\triangle P_n P_{n+1} P_{n+2})}{A(\triangle P_{n+1} P_{n+2} P_{n+3})}$ is constant. [12]

Solution: Let $f(x) = x^5$. For $x_1 > 0$, let $P_1 = (x_1, f(x_1))$. Draw a tangent at the point P_1 and let it meet the graph again at point P_2. Recursively P_{n+1} is defined. We now try to calculate P_2 in terms of P_1. Tangent at P_1 is given by $y - y_1 = 5x_1^4(x - x_1)$ i.e. $y = 5x_1^4 x - 4x_1^5$. This cuts the curve $y = x^5$ at $x = x_2,\ x_2 \neq x_1$. Hence

$$x_2^5 - 5x_1^4 x_2 + 4x_1^5 = 0.$$

This is a *homogeneous* equation in x_1, x_2. So put $x_2 = kx_1,\ k \neq 1$, then $x_1^5 k^5 - 5kx_1^5 + 4x_1^5 = 0$. This implies $k^5 - 5k + 4 = 0$ i.e. $(k-1)^2(k^3 + 2k^2 + 3k + 4) = 0$. Since $k \neq 1$, $k^3 + 2k^2 + 3k + 4 = 0$. Observe that this cubic equation has one real and two complex roots. ($g'(x) = 3k^2 + 4k + 3 \neq 0$.) The real root k must be negative. [5]

If $x_3 = \ell x_2$, then by similar argument, we see that ℓ is again the above negative root k of $k^3 + 2k^2 + 3k + 4 = 0$. Thus $x_3 = kx_2 = k^2 x_1$. Hence by induction, $x_{n+1} = k^n x_1$ for all $n \geq 1$. [2]
We now calculate $A(\triangle P_n P_{n+1} P_{n+2})$

$$A(\triangle P_n P_{n+1} P_{n+2}) = \frac{1}{2}\det\begin{pmatrix} x_n & x_n^5 & 1 \\ x_{n+1} & x_{n+1}^5 & 1 \\ x_{n+2} & x_{n+2}^5 & 1 \end{pmatrix}$$
$$= \frac{1}{2}\det\begin{pmatrix} k^{n-1}x_1 & k^{5n-5}x_1^5 & 1 \\ k^n x_1 & k^{5n}x_1^5 & 1 \\ k^{n+1}x_1 & k^{5n+5}x_1^5 & 1 \end{pmatrix}$$

$$= \frac{k^{n-1}x_1k^{5n-5}x_1^5}{2}\det\begin{pmatrix}1 & 1 & 1\\ k & k^5 & 1\\ k^2 & k^{10} & 1\end{pmatrix}$$

$= k^{6n-6}x_1^6D$, where $D = \frac{1}{2}\det\begin{pmatrix}1 & 1 & 1\\ k & k^5 & 1\\ k^2 & k^{10} & 1\end{pmatrix} \neq 0$ since $k \neq -1$.

Then $A(\triangle P_{n+1}P_{n+2}P_{n+3}) = k^{6n}x_1^6D$.

Hence the ratio $\frac{A(\triangle P_nP_{n+1}P_{n+2})}{A(\triangle P_{n+1}P_{n+2}P_{n+3})} = \frac{1}{k^6}$ is constant. [5]

4. Let $p(x)$ be a polynomial with positive integer coefficients. You can ask the question: What is $p(n)$ for any positive integer n? What is the minimum number of questions to be asked to determine $p(x)$ completely? Justify. [13]

Solution: The minimum number of questions needed is 2. For this, let $p(x)$ be a polynomial with positive integer coefficients say, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$. We can ask the question: what is $p(1)$? Let $p(1) = N$. [3]

Here $N = a_0 + a_1 + a_2 + \cdots + a_k > a_i,\ \forall i$ and N is a known number.

Also, what is $p(N)$? So $p(N) = a_0 + a_1N + a_2N^2 + \cdots + a_kN^k$ is a known number. [4]

Now express $p(N)$ to base N, then i^{th} digit gives $a_i,\ \forall i$ because $a_i < N,\ \forall i$. Thus $p(x)$ is determined. [6]

Note that asking only one question i.e. asking for the value $p(n)$ for a particular choice of n, is not sufficient to determine the polynomial $p(x)$.

Example. Suppose $p(1) = 9$ and $p(9) = 193$. Now we express 193 to base 9 :

$$193 = 21(9) + 4, \quad 21 = 2(9) + 3, \quad 2 = 0(9) + 2.$$

So the remainders are, starting with the last, $2, 3, 4$. So $193 = 2(9^2)+3(9)+4(9^0) = (234)_9$. So $a_2 = 2, a_1 = 3, a_0 = 4$ and $p(x) = 4 + 3x + 2x^2$.

MADHAVA MATHEMATICS COMPETITION

(A Mathematics Competition for Undergraduate Students)

Organized by

Department of Mathematics, S. P. College, Pune

and

Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date : 4 / 1 / 2015 **Max. Marks : 100**

Time : 12.00 noon to 3.00 p.m.

N. B. : Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. How many five digit positive integers that are divisible by 3 can be formed using the digits $0, 1, 2, 3, 4$ and 5, without any of the digits getting repeated?
   A) 216 B) 96 C) 120 D) 625 .

2. If $I = \int_0^1 \frac{1}{1+x^8}\, dx$, then
   A) $I < \frac{1}{2}$ B) $I < \frac{\pi}{4}$ C) $I > \frac{\pi}{4}$ D) $I = \frac{\pi}{4}$.

3. Find a and b so that $y = ax + b$ is a tangent line to the curve $y = x^2 + 3x + 2$ at $x = 3$.
   A) $a = 9, b = -7$ B) $a = 3, b = -2$ C) $a = -9, b = 7$ D) $a = -3, b = 2$.

4. Suppose p is a prime number. The possible values of gcd of $p^3 + p^2 + p + 11$ and $p^2 + 1$ are
   A) 1,2,5 B) 2,5,10 C) 1,5,10 D) 1,2,10.

5. Consider all 2×2 matrices whose entries are distinct and belong to $\{1, 2, 3, 4\}$. The sum of determinants of all such matrices is
   A) 4! B) 0 C) negative D) odd.

6. Choose the correct alternative:
   A) The Taylor series of $\sin\frac{1}{x}$ about $x = \frac{2}{\pi}$ does not exist.
   B) The coefficient of $\left(x - \frac{2}{\pi}\right)^2$ in the Taylor series of $\sin\frac{1}{x}$ about $x = \frac{2}{\pi}$ is $\frac{-\pi^4}{32}$.
   C) The Taylor series of $\sin\frac{1}{x}$ about $x = \frac{2}{\pi}$ has negative powers of x.
   D) The coefficient of $\left(x - \frac{2}{\pi}\right)^2$ in the Taylor series of $\sin\frac{1}{x}$ about $x = \frac{2}{\pi}$ is 0.

7. Consider all right circular cylinders for which the sum of the height and circumference of the base is 30 cm. The radius of the one with maximum volume is
   A) 3 B) 10 C) $\frac{10}{\pi}$ D) $\frac{\pi}{10}$.

8. In how many ways can you express $2^3 3^5 5^7 7^{11}$ as a product of two numbers, ab, where $\gcd(a, b) = 1$ and $1 < a < b$?
   A) 5 B) 6 C) 7 D) 8.

9. The value of $\int_a^b \sin x\, dx$ is
   A) $(b-a)\sin c$ B) $(b-a)\cos c$ C) $\frac{\sin c}{b-a}$ D) $\frac{\cos c}{b-a}$
   for some real number c such that $a \le c \le b$.

10. Suppose a, b, c are three distinct integers from 2 to 10 (both inclusive). Exactly one of ab, bc and ca is odd and abc is a multiple of 4. The arithmetic mean of a and b is an integer and so is the arithmetic mean of a, b and c. How many such (unordered) triplets are possible?
    A) 4 B) 5 C) 6 D) 7.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $P(x) = \sum_{r=0}^{n} c_r x^r$ be a polynomial with real coefficients with $c_0 > 0$ and $\sum_{r=0}^{[n/2]} \frac{c_{2r}}{2r+1} < 0$. Prove that P has root in $(-1, 1)$.

2. If $|z_1| = |z_2| = |z_3| > 0$ and $z_1 + z_2 + z_3 = 0$. then show that the points representing the complex numbers z_1, z_2, z_3 form an equilateral triangle.

3. If $1, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are n^{th} roots of unity, prove that
$$\frac{1}{2-\alpha_1} + \frac{1}{2-\alpha_2} + \cdots + \frac{1}{2-\alpha_{n-1}} = \frac{(n-2)2^{n-1}+1}{2^n - 1}.$$

4. Let $f(x)$ be a monic polynomial of degree 4 such that $f(1) = 10, f(2) = 20, f(3) = 30$. Find $f(12) + f(-8)$.

5. Find all solutions (a, b, c, n) in positve integers for the equation $2^n = a! + b! + c!$.

Part III

1. Suppose the polynomials f and g have the same roots and $\{x \in \mathbb{C} : f(x) = 2015\} = \{x \in \mathbb{C} : g(x) = 2015\}$, then show that $f = g$. [13]

2. Give an example of a function which is continuous at exactly two points and differentiable at exactly one of them. Justify your answer. [13]

3. Let A be any $m \times n$ matrix whose entries are positive inegers. A step consists of transforming the matrix either by multiplying every entry of *a row* by 2 or subtracting 1 from every entry of *a column*. Can you transform A into the zero matrix in finitely many steps? Justify your answer. [12]

4. Let S be the set of positive integers that do not have zero in their decimal representation. Thus $S = \{1, 2, 3, \cdots, 9, 11, 12, \cdots, 19, 21, \cdots, 99, 111, \cdots\}$.
Show that the series $\sum_{n \in S} \frac{1}{n}$ converges. [12]

MADHAVA MATHEMATICS COMPETITION, January 4, 2015

Part I

N.B. Each question in Part I carries 2 marks.

1. How many five digit positive integers that are divisible by 3 can be formed using the digits $0, 1, 2, 3, 4$ and 5, without any of the digits getting repeated?

A) 216 B) 96 C) 120 D) 625 .

Answer: A) Numbers of the form *abcde* with distinct digits from the set $\{0, 1, 2, 3, 4, 5\}$ such that $a \neq 0$ and $3 \mid (a + b + c + d + e)$. Since $3|(1 + \cdots + 5) = 15$, there are $5! = 120$ such numbers with no digit zero. If 0 is included, then 3 must be excluded; so for $a = 1$, 4! numbers like 10245; for $a = 2$, 4! numbers like 20145; for $a = 4$, 4! numbers like 40245; for $a = 5$, 4! numbers like 50124. So $120 + 96 = 216$ in all.

2. If $I = \displaystyle\int_0^1 \frac{1}{1+x^8}\, dx$, then

A) $I < \dfrac{1}{2}$ B) $I < \dfrac{\pi}{4}$ C) $I > \dfrac{\pi}{4}$ D) $I = \dfrac{\pi}{4}$.

Answer: C) Note that $0 < x < 1 \Rightarrow x^8 < x^2 \Rightarrow 1 + x^8 < 1 + x^2 \Rightarrow 1/(1 + x^8) > 1/(1 + x^2)$. So $I > \int_0^1 1/(1 + x^2)dx = \pi/4$.

3. Find a and b so that $y = ax + b$ is a tangent line to the curve $y = x^2 + 3x + 2$ at $x = 3$.

A) $a = 9, b = -7$ B) $a = 3, b = -2$ C) $a = -9, b = 7$ D) $a = -3, b = 2$.

Answer: A) Since for $f(x) = x^2 + 3x + 2$, $f'(x) = 2x + 3$, the tangent at $x = 3$ is $y - f(3) = f'(3)(x - 3)$ i.e. $y - 20 = 9(x - 3)$ or $y = 9x - 7$.

4. Suppose p is a prime number. The possible values of gcd of $p^3 + p^2 + p + 11$ and $p^2 + 1$ are

A) 1,2,5 B) 2,5,10 C) 1,5,10 D) 1,2,10.

Answer: B) Let $a = p^3 + p^2 + p + 11$, $b = p^2 + 1$, and $d =$ gcd of a and b. Then since $a = b(p + 1) + 10$, d divides 10. So $d = 1, 2, 5$ or 10. But for $p = 2$, we get $d = 5$ and for odd p, both a, b are even, hence d is even. So $d \neq 1$.

5. Consider all 2×2 matrices whose entries are distinct and belong to $\{1, 2, 3, 4\}$. The sum of determinants of all such matrices is

A) 4! B) 0 C) negative D) odd.

Answer: B) There are in all $4! = 24$ matrices. These can be taken in pairs like $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right)$ where B is obtained by interchanging the rows of A; so $\det A = -\det B$ or $\det A + \det B = 0$.

6. Choose the correct alternative:

A) The Taylor series of $\sin\dfrac{1}{x}$ about $x = \dfrac{2}{\pi}$ does not exist.

B) The coefficient of $\left(x - \dfrac{2}{\pi}\right)^2$ in the Taylor series of $\sin\dfrac{1}{x}$ about $x = \dfrac{2}{\pi}$ is $\dfrac{-\pi^4}{32}$.

C) The Taylor series of $\sin\dfrac{1}{x}$ about $x = \dfrac{2}{\pi}$ has negative powers of x.

D) The coefficient of $\left(x - \dfrac{2}{\pi}\right)^2$ in the Taylor series of $\sin\dfrac{1}{x}$ about $x = \dfrac{2}{\pi}$ is 0.

Answer: B) For $f(x) = \sin(1/x)$, $f'(x) = -(1/x^2)\cos(1/x)$ and $f''(x) = (2/x^3)\cos(1/x) -$

$(1/x^4)\sin(1/x)$. So $f(2/\pi) = 1$, $f'(2/\pi) = 0$, and $f''(2/\pi) = -\pi^4/16$. So the Taylor series is

$$\begin{aligned} f(x) &= f(2/\pi) + f'(2/\pi)(x - 2/\pi) + \frac{1}{2!}f''(2/\pi)(x - 2/\pi)^2 + \cdots \\ &= 1 + \frac{1}{2}\left[\frac{-\pi^4}{16}\right](x - 2/\pi)^2 + \cdots . \end{aligned}$$

7. Consider all right circular cylinders for which the sum of the height and circumference of the base is 30 cm. The radius of the one with maximum volume is

A) 3 B) 10 C) $\frac{10}{\pi}$ D) $\frac{\pi}{10}$.

Answer: C) Let x and h be the radius of base and height of a cylinder. Then the circumference is $2\pi x + h$. So $2\pi x + h = 30$ by data. So the volume is $v(x) = \pi x^2 h = \pi x^2(30 - 2\pi x) = 2\pi(15x^2 - \pi x^3)$. So $v'(x) = 2\pi(30x - 3\pi x^2)$, $v''(x) = 2\pi(30 - 6\pi x)$. Now $v'(x) = 0$ gives $x = 10/\pi$ as the only non-zero critical value of x and $v''(10/\pi) = 2\pi(30 - 60) < 0$. So v is maximum at $x = 10/\pi$.

8. In how many ways can you express $2^3 3^5 5^7 7^{11}$ as a product of two numbers, ab, where $\gcd(a, b) = 1$ and $1 < a < b$?

A) 5 B) 6 C) 7 D) 8.

Answer: C) Let $x = 2^3$, $y = 3^5$, $z = 5^7$, $w = 7^{11}$. Then the 7 ways are $(x)(yzw)$, $(y)(xzw)$ $(z)(xyw)$, $(w)(xyz)$; $(xy)(zw)$, $(xz)(yw)$, $(yz)(xw)$.

9. The value of $\displaystyle\int_a^b \sin x \, dx$ is

A) $(b - a)\sin c$ B) $(b - a)\cos c$ C) $\frac{\sin c}{b - a}$ D) $\frac{\cos c}{b - a}$

for some real number c such that $a \le c \le b$.

Answer: A) Since the function $f(x) = \sin x$ is continuous in $[a, b]$, by the mean-value theorem there is $c \in [a, b]$ such that the integral $= (b - a)f(c)$.

10. Suppose a, b, c are three distinct integers from 2 to 10 (both inclusive). Exactly one of ab, bc and ca is odd and abc is a multiple of 4. The arithmetic mean of a and b is an integer and so is the arithmetic mean of a, b and c. How many such (unordered) triplets are possible?

A) 4 B) 5 C) 6 D) 7.

Answer: A) Since exactly one of ab, bc and ca is odd and $4|abc$, two of the numbers must be odd and the remaining must be a multiple of 4. Since the A.M., $(a + b)/2$, of a, b is an integer, $2|(a + b)$ so that a, b have the same parity so that both are odd by the above. Since the A.M., $(a + b + c)/3$, of a, b, c is an integer, $3|(a + b + c)$. So the only triplets are $(a, b, c) = (3, 5, 4)$; $(3, 7, 8)$; $(5, 9, 4)$ and $(7, 9, 8)$.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $P(x) = \displaystyle\sum_{r=0}^{n} c_r x^r$ be a polynomial with real coefficients with $c_0 > 0$ and $\displaystyle\sum_{r=0}^{[n/2]} \frac{c_{2r}}{2r + 1} < 0$. Prove that P has root in $(-1, 1)$.

Solution : $P(0) = c_0 > 0$. **[1 mk]**

Note that

$$\begin{aligned} I = \int_{-1}^{1} P(x)dx &= \int_{-1}^{0} P(x)dx + \int_{0}^{1} P(x)dx \quad (\text{Put } x = -t) \\ &= \int_{1}^{0} P(-t)(-dt) + \int_{0}^{1} P(x)dx \\ &= \int_{0}^{1} [P(-x) + P(x)]dx \\ &= 2\int_{0}^{1} \sum_{r=0}^{[n/2]} c_{2r}x^{2r}dx = 2\sum_{r=0}^{[n/2]} \frac{c_{2r}}{2r+1}. \end{aligned}$$ **[3** mks]

By data, $\displaystyle\sum_{r=0}^{[n/2]} \frac{c_{2r}}{2r+1} < 0$, so that $I < 0$. Now P is a continuous function on $[-1, 1]$ and its integral I over $[-1, 1]$ is *negative*. Hence there is $k \in [-1, 1]$ such that $P(k) < 0$. (Recall: If f is a continuous function on $[a, b]$ $(a < b)$ such that $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq 0$.) Also, $P(0) = c_0 > 0$. Therefore, since P is a continuous function on $[-1, 1]$, there is a number α between k and 0 such that $P(\alpha) = 0$. **[2 mks]**

2. If $|z_1| = |z_2| = |z_3| > 0$ and $z_1 + z_2 + z_3 = 0$, then show that the points representing the complex numbers z_1, z_2, z_3 form an equilateral triangle.
Solution: There are unique numbers $\theta_1, \theta_2, \theta_3$ in $[0, 2\pi)$ such that

$$z_k = r(\cos\theta_k + i\sin\theta_k), \quad k = 1, 2, 3, \tag{1}$$

where $r = |z_1| = |z_2| = |z_3|$. Then if A, B, C are the points represented by the numbers z_1, z_2, z_3 respectively, then they lie on the circle $|z| = r$ with origin O as its centre. The condition $z_1 + z_2 + z_3 = 0$ gives

$$r[(\cos\theta_1 + \cos\theta_2 + \cos\theta_3) + i(\sin\theta_1 + \sin\theta_2 + \sin\theta_3)] = 0. \quad \textbf{[1} \text{ mk]}$$

As $r > 0$, this gives $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$ and $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0$. Hence

$$\cos\theta_1 + \cos\theta_2 = -\cos\theta_3, \quad \sin\theta_1 + \sin\theta_2 = -\sin\theta_3. \quad \textbf{[2} \text{ mks]}$$

Squaring and adding we get $1 + 1 - 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = 1$ or $\cos(\theta_2 - \theta_1) = -1/2$. So $\angle AOB = \theta_2 - \theta_1 = 2\pi/3$. Similarly $\angle BOC = \angle COA = 2\pi/3$. Hence the chords AB, BC, CA subtend the same angle, $2\pi/3$, at the centre and so $AB = BC = CA$. So $\triangle ABC$ is equilateral. **[3 mks]**

3. If $1, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are n^{th} roots of unity, prove that

$$\frac{1}{2-\alpha_1} + \frac{1}{2-\alpha_2} + \cdots + \frac{1}{2-\alpha_{n-1}} = \frac{(n-2)2^{n-1}+1}{2^n - 1}.$$

Solution: First method: Let $f(x) = x^n - 1$ so that $\alpha_0 = 1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the n roots of $f(x) = 0$ and of these $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are non-real for $n \geq 3$. For a fixed r,

$0 \le r \le n-1$, we have the identity

$$\begin{aligned}\frac{x^n-1}{x-\alpha_r} &= \frac{x^n-\alpha_r^n}{x-\alpha_r}\\ &= x^{n-1}+x^{n-2}\alpha_r+x^{n-3}\alpha_r^2+\cdots+x\alpha_r^{n-2}+\alpha_r^{n-1}. \qquad \textbf{[1 mk]}\end{aligned}$$

Putting $x=2$ and summing over r we get

$$\begin{aligned}&\sum_{r=0}^{n-1}\frac{2^n-1}{2-\alpha_r}\\ &=\sum_r 2^{n-1}+2^{n-2}\sum_r\alpha_r+\cdots+2\sum_r\alpha_r^{n-2}+\sum_r\alpha_r^{n-1}. \qquad (1)\end{aligned}$$

Now the nth roots of unity are $\alpha_r = e^{(2\pi i)r/n}$, $r=0,1,\ldots,n-1$. Thus $\alpha_r=\alpha_1^r$, $r=0,1,\ldots,n-1$. For a fixed j, $1\le j\le n-1$, we have

$$\sum_{r=0}^{n-1}\alpha_r^j=\sum_{r=0}^{n-1}(\alpha_1^r)^j=\sum_{r=0}^{n-1}(\alpha_1^j)^r=\frac{1-(\alpha_1^j)^n}{1-\alpha_1^j}=0 \qquad \textbf{[3 mks]}$$

since $(\alpha_1^j)^n=(\alpha_1^n)^j=1^j=1$. So substituting in (1) we have

$$\begin{aligned}&(2^n-1)\sum_{r=0}^{n-1}\frac{1}{2-\alpha_r}=n2^{n-1}\\ \text{or}\quad &1+\sum_{r=1}^{n-1}\frac{1}{2-\alpha_r}=\frac{n2^{n-1}}{2^n-1} \quad (\text{as } \alpha_0=1)\\ \text{or}\quad &\sum_{r=1}^{n-1}\frac{1}{2-\alpha_r}=\frac{n2^{n-1}}{2^n-1}-1=\frac{(n-2)2^{n-1}+1}{2^n-1}. \qquad \textbf{[2 mks]}\end{aligned}$$

Second method: Let $f(x)=x^n-1$ so that $\alpha_0=1,\alpha_1,\alpha_2,\ldots,\alpha_{n-1}$ are the n roots of $f(x)=0$. Hence we have the identity

$$f(x)=(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1}). \qquad \textbf{[2 mks]}$$

Taking logarithm on both the sides and differentiating we get the identity

$$\frac{nx^{n-1}}{x^n-1}=\frac{1}{x-\alpha_0}+\frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\cdots+\frac{1}{x-\alpha_{n-1}},$$

for x different from all the α_i. So putting $x=2$ we get,

$$\begin{aligned}&\frac{1}{2-\alpha_0}+\frac{1}{2-\alpha_1}+\frac{1}{2-\alpha_2}+\cdots+\frac{1}{2-\alpha_{n-1}}=\frac{n2^{n-1}}{2^n-1}, \quad (\alpha_0=1)\\ &\frac{1}{2-\alpha_1}+\frac{1}{2-\alpha_2}+\cdots+\frac{1}{2-\alpha_{n-1}}=\frac{n2^{n-1}}{2^n-1}-1=\frac{(n-2)2^{n-1}+1}{2^n-1}. \qquad \textbf{[4 mks]}\end{aligned}$$

4. Let $f(x)$ be a monic polynomial of degree 4 such that $f(1) = 10, f(2) = 20, f(3) = 30$. Find $f(12) + f(-8)$.
Solution: First method: Note that $f(1) = 10$ means that the remainder is 10 when $f(x)$ is divided by $x - 1$. Similarly, the remainder is 10×2 when $f(x)$ is divided by $x - 2$ and the remainder is 10×3 when $f(x)$ is divided by $x - 3$. Hence we can take the degree 4 monic polynomial $f(x)$ to be $f(x) = (x - 1)(x - 2)(x - 3)(x - k) + 10x$, where k is a parameter. Then **[4** mks]

$$f(12) = 11 \cdot 10 \cdot 9(12 - k) + 120 = 990(12 - k) + 120,$$
$$f(-8) = (-9)(-10)(-11)(-8 - k) - 80 = -990(-8 - k) - 80.$$

Adding, $f(12) + f(-8) = 990(12 - k + 8 + k) + 40 = 990(20) + 40 = 19840$. **[2** mks]

Second method: Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ so that by the data we have the equations

$$1 + a + b + c + d = 10,$$
$$16 + 8a + 4b + 2c + d = 20,$$
$$81 + 27a + 9b + 3c + d = 30.$$ **[2** mks]

Reducing this system we get

$$a + b + c + d = 9,$$
$$36b + 54c + 63d = 612,$$
$$6c + 11d = 24.$$

Solving, we get $c = 4 - \frac{11}{6}d$, $b = 11 + d$, $a = -6 - \frac{1}{6}d$. So **[3 mks]**

$$f(x) = x^4 + (-6 - \frac{1}{6}d)x^3 + (11 + d)x^2 + (4 - \frac{11}{6}d)x + d$$
$$= x^4 - 6x^3 + 11x^2 + 4x + d\left[-\frac{1}{6}x^3 + x^2 - \frac{11}{6}x + 1\right].$$

Hence $f(12) + f(-8) = 12000 - 165d + 7840 + 165d = 19840$. **[1 mk]**

5. Find all solutions (a, b, c, n) in positve integers for the equation $2^n = a! + b! + c!$.
Solution: Let $a \leq b \leq c$. Then $N = 1 + \frac{b!}{a!} + \frac{c!}{a!}$ is an integer since $\frac{b!}{a!}$ and $\frac{c!}{a!}$ are both integers. Therefore, since $2^n = a! \cdot N$, we see that $a!$ divides 2^n so that we must have $a = 1$ or $a = 2$.

Let $a = 1$. Then $2^n - 1 = b! + c!$ where left side is odd. So exactly one of $b!$ and $c!$ must be odd. But $k!$ is odd only when $k = 1$. Therefore $b = 1$ since $1 \leq b \leq c$. So $2^n - 2 = c!$ or $2(2^{n-1} - 1) = c!$ and left side is not divisible by 4. So $c \leq 3$, and $(a, b, c, n) = (1, 1, 2, 2), (1, 1, 3, 3)$ are the corresponding solutions. **[4 mks]**

Let $a = 2$. Then $2 \leq b \leq c$ and $2(2^{n-1} - 1) = b!(1 + \frac{c!}{b!})$. So as before, $b = 2, 3$. Let $b = 2$. Then $2 \leq c$ and $4(2^{n-2} - 1) = c!$ which is not possible as 8 divides $c!$ for $c \geq 4$ and 4 does not divide 2! and 3!. Let $b = 3$. Then $3 \leq c$ and $8(2^{n-3} - 1) = c!$ so that $4 \leq c \leq 5$ as 16 divides $c!$ for $c \geq 6$. Here $(a, b, c, n) = (2, 3, 4, 5), (2, 3, 5, 7)$ are the corresponding solutions.

Thus the only possibilities are $2^2 = 1! + 1! + 2!$, $2^3 = 1! + 1! + 3!$, $2^5 = 2! + 3! + 4!$ and $2^7 = 2! + 3! + 5!$. **[2 mks]**

Part III

1. Suppose the polynomials f and g have the same roots and $\{x \in \mathbb{C} \,:\, f(x) = 2015\} = \{x \in \mathbb{C} \,:\, g(x) = 2015\}$, then show that $f = g$. [13]

Solution: The polynomials f and g have *the same* roots. Let these roots be $\alpha_1, \ldots, \alpha_n$, and let the multiplicities of these be $a_1, \ldots, a_n$ for f and $b_1, \ldots, b_n$ for g respectively. Then the degree of f is $N_1 = a_1 + \cdots + a_n$ and the degree of g is $N_2 = b_1 + \cdots + b_n$, and we have

$$\begin{aligned} f(x) &= (x-\alpha_1)^{a_1}(x-\alpha_2)^{a_2}\cdots(x-\alpha_n)^{a_n}, \\ g(x) &= (x-\alpha_1)^{b_1}(x-\alpha_2)^{b_2}\cdots(x-\alpha_n)^{b_n}, \\ \text{where} \quad & N_1 \geq N_2, \quad \text{say.} \end{aligned} \tag{1}$$

[2 mks]

Next, by data the polynomials $F(x) = f(x) - 2015$ and $G(x) = g(x) - 2015$ also have *the same* roots, say $\beta_1, \ldots, \beta_m$. Let their multiplicities be $a'_1, \ldots, a'_m$ for F and $b'_1, \ldots, b'_m$ for G respectively. Then degree of F = degree of $f = N_1 = a'_1 + \cdots + a'_m$ and degree of G = degree of $g = N_2 = b'_1 + \cdots + b'_m$, and we have

$$\begin{aligned} F(x) &= (x-\beta_1)^{a'_1}(x-\beta_2)^{a'_2}\cdots(x-\beta_m)^{a'_m}, \\ G(x) &= (x-\beta_1)^{b'_1}(x-\beta_2)^{b'_2}\cdots(x-\beta_m)^{b'_m}. \end{aligned}$$

Note that the sets $S = \{\alpha_1, \ldots, \alpha_n\}$ and $T = \{\beta_1, \ldots, \beta_m\}$ are *disjoint*.

Let, if possible, $f \neq g$ i.e. let $f - g$ be a *non-zero* polynomial. Then as $\alpha_1, \ldots, \alpha_n$ are roots of both f and g, they are roots of $f - g$ also. Similarly, $\beta_1, \ldots, \beta_m$ are roots of $F - G = f - g$. Therefore, since the sets S, T are disjoint, we have

$$f - g = (x-\alpha_1)\cdots(x-\alpha_n)(x-\beta_1)\cdots(x-\beta_m)H(x), \tag{5 mks}$$

for some polynomial $H(x)$. Hence $f - g$ is a *non-constant* polynomial of degree say K, and

$$K \geq n + m. \tag{2}$$

To obtain a contradiction we consider the multiplicities of the roots of the derivative f'. Now α_1 is a root of f with multiplicity a_1 means that

$$f(x) = (x-\alpha_1)^{a_1}\phi(x)$$

where $\phi(\alpha_1) \neq 0$. This gives $f'(x) = a_1(x-\alpha_1)^{a_1-1}\phi(x) + (x-\alpha_1)^{a_1}\phi'(x)$. So $f'(x) = (x-\alpha_1)^{a_1-1}\psi(x)$ where $\psi(x) = a_1\phi(x) + (x-\alpha_1)\phi'(x)$. Therefore, since $\psi(\alpha_1) = a_1\phi(\alpha_1) \neq 0$, we see that α_1 is a root of f' with multiplicity $a_1 - 1$. Thus $\alpha_1, \ldots, \alpha_n$ are roots of f' with multiplicities $a_1 - 1, \ldots, a_n - 1$. Similarly, since $f' = F'$, it follows that $\beta_1, \ldots, \beta_m$ are also roots of f' with multiplicities $a'_1 - 1, \ldots, a'_m - 1$. Hence the degree of f' namely, $N_1 - 1$, is such that

$$\begin{aligned} N_1 - 1 &\geq \sum_{i=1}^{n}(a_i - 1) + \sum_{j=1}^{m}(a'_j - 1) = N_1 - n + N_1 - m \\ \text{or} \quad & n + m - 1 \geq N_1. \end{aligned} \tag{3}$$

By (1), the degree K of $f-g$ satisfies $K \leq N_1$. So by (2) and (3), we get $n+m \leq K \leq N_1 \leq n+m-1$ or $n+m \leq n+m-1$, contradiction. So $f=g$. **[6 mks]**

2. Give an example of a function which is continuous at exactly two points and differentiable at exactly one of them. Justify your answer. [13]
Solution Define the function $f : \mathbb{R} \to \mathbb{R}$ thus:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ x^3 & \text{if } x \text{ is irrational} \end{cases}$$ **[5** mks]

We show that f is continuous only at 0 and 1, and differentiable only at 0. For this, consider a real number a. Then as $x \to a$ through rational values, $f(x) = x^2 \to a^2$, and as $x \to a$ through irrational values, $f(x) = x^3 \to a^3$. So the limit $\lim_{x\to a} f(x)$ will exist if and only if the above two limits are equal i.e. if and only if $a^2 = a^3$ i.e. $a^2(a-1) = 0$ i.e. $a = 0$ or $a = 1$. Thus f is continuous at 0 since $\lim f(x) = \lim x^2 = 0 = f(0)$. Similarly, f is continuous at 1. But when $a \neq 0, 1$, $\lim_{x\to a} f(x)$ does not exist; so f is discontinuous at a. **[4 mks]**

Next, let $g(x) = [f(x) - f(a)]/(x-a)$. Let a be rational. As $x \to a$ through irrational values, $\lim g(x) = \lim\{[x^3 - a^2]/(x-a)\}$ is not finite if $\lim[x^3 - a^2] \neq 0$ i.e. if $a^3 \neq a^2$ i.e. if $a \notin \{0, 1\}$. Hence $f'(a)$ does not exist (finitely) if $a \notin \{0, 1\}$. Let $a = 0$. Then $\lim_{x\to 0} g(x) = \lim_{x\to 0}[f(x)/x] = 0$. So $f'(0)$ exists and is 0.

But as $x \to 1$ through rational values, $\lim g(x) = \dfrac{x^2-1}{x-1} = 2$, while as $x \to 1$ through irrational values, $\lim g(x) = \lim \dfrac{x^3-1}{x-1} = 3$. Hence $f'(1)$ does not exist.

Let a be irrational. As $x \to a$ through rational values,

$$\lim g(x) = \lim\{[x^2 - a^3]/(x-a)\}$$

is not finite if $\lim[x^2 - a^3] \neq 0$ i.e. if $a^2 \neq a^3$ i.e. if $a \notin \{0, 1\}$. Hence $f'(a)$ does not exist. **[4 mks]**

3. Let A be any $m \times n$ matrix whose entries are positive inegers. A step consists of transforming the matrix either by multiplying every entry of *a row* by 2 or subtracting 1 from every entry of *a column*. Can you transform A into the zero matrix in finitely many steps? Justify your answer. [12]

Solution Yes, one method is as follows: Let $A = [a_{ij}]$ be the matrix. Let m be the minimum element in the first column C_1. In fact, let m occur s times i.e. let $m = a_{i_1 1} = \cdots = a_{i_s 1}$. We may *assume* that $m = 1$. For if $m \geq 2$, subtract 1 from each element of C_1 $m-1$ times so that the minimum element in C_1 is 1. **[4 mks]**

Multiply each of the s rows $i_1, i_2, \ldots, i_s$ of A by 2. This forces the minimum element in C_1 to be 2. Subtract 1 from each element of C_1. The effect of these steps on C_1 is this: the s elements $a_{i_1 1}, a_{i_2 1}, \ldots, a_{i_s 1}$ of C_1 are still equal to 1, but the remaining elements of C_1 have all become *smaller* though they are all still ≥ 1. Hence in a finite number of steps *all* elements of C_1 will become 1. Then subtracting 1 from each element of C_1 makes C_1 a column of zeros.

Next make the second column C_2 a column of zeros as in the above. Note that the operations on C_2 have no effect on C_1 and C_1 *remains a column of zeros.* Hence in a finite number of steps A becomes the zero matrix. **[8 mks]**

4. Let S be the set of positive integers that do not have zero in their decimal representation. Thus $S = \{1, 2, 3, \cdots, 9, 11, 12, \cdots, 19, 21, \cdots, 99, 111, \cdots\}$.
Show that the series $\sum_{n\in S} \frac{1}{n}$ converges. [12]

Solution: Let A denote the series $\sum_{n\in S} \frac{1}{n}$. Let t_1 denote the sum of the first 9 terms of A, namely $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{9}$. Then $t_1 < 9$ since each of these 9 terms is ≤ 1. **[2 mks]**

Next, let t_2 denote the sum of the *next* $81 = 9^2$ terms of A, namely

$$\begin{array}{c} \frac{1}{11}, \frac{1}{12}, \ldots, \frac{1}{19}, \\ \frac{1}{21}, \frac{1}{22}, \ldots, \frac{1}{29}, \\ \cdots \quad \cdots \quad \cdots \\ \frac{1}{91}, \frac{1}{92}, \ldots, \frac{1}{99}. \end{array}$$ **[3** mks]

Then $t_2 < \frac{1}{10} \cdot 9^2$ since each of these 9^2 terms is $\leq \frac{1}{11} < \frac{1}{10}$.

Similarly, let t_3 denote the sum of the *next* $729 = 9^3$ terms of A, namely

$$\begin{array}{c} \frac{1}{111}, \frac{1}{112}, \ldots, \frac{1}{119}, \\ \frac{1}{221}, \frac{1}{222}, \ldots, \frac{1}{229}, \\ \cdots \quad \cdots \quad \cdots \\ \frac{1}{991}, \frac{1}{992}, \ldots, \frac{1}{999}. \end{array}$$ **[3** mks]

Then $t_3 < \frac{1}{10^2} \cdot 9^3$ since each of these 9^3 terms is $\leq \frac{1}{111} < \frac{1}{100}$.

Proceeding in this way, if s_n is the sum of the first n terms of A, we take m to be the smallest positive integer such that $n \leq 9 + 9^2 + \cdots + 9^m = 9(9^m - 1)/8 = N$, say. Then since the terms are positive and since the first n terms of A are included in the first N terms of A, we see that

$$\begin{aligned} s_n < s_N &= 9 + \frac{1}{10} \cdot 9^2 + \frac{1}{10^2} \cdot 9^3 + \cdots + \frac{1}{10^{m-1}} \cdot 9^m \\ &= 9 \left[1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \cdots + \left(\frac{9}{10}\right)^{m-1} \right] \\ &= 9 \cdot \frac{1 - (9/10)^m}{1 - (9/10)} < 90. \end{aligned}$$

So the partial sums of the positive term series A are bounded above. Hence series A is convergent. **[4 mks]**

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 08/01/2017 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The number $\sqrt{2}e^{i\pi}$ is:
   A) a rational number.
   B) an irrational number.
   C) a purely imaginary number.
   D) a complex number of the type $a + ib$ where a, b are non-zero real numbers.

2. Let P= $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The rank of P^4 is: A) 1 B) 2 C) 3 D) 4.

3. Let $y_1(x)$ and $y_2(x)$ be the solutions of the differentiable equation $\frac{dy}{dx} = y + 17$ with initial conditions $y_1(0) = 0, y_2(0) = 1$. Which of the following statements is true?
   A) y_1 and y_2 will never intersect.
   B) y_1 and y_2 will intersect at $x = e$.
   C) y_1 and y_2 will intersect at $x = 17$.
   D) y_1 and y_2 will intersect at $x = 1$.

4. Suppose f and g are differentiable functions and $h(x) = f(x)g(x)$. Let $h(1) = 24$, $g(1) = 6$, $f'(1) = -2$, $h'(1) = 20$. Then the value of $g'(1)$ is
   A) 8 B) 4 C) 2 D) 16.

5. In how many regions is the plane divided when the following equations are graphed, not considering the axes? $y = x^2$, $y = 2^x$
   A) 3 B) 4 C) 5 D) 6.

6. For $0 \leq x < 2\pi$, the number of solutions of the equation $\sin^2 x + 3\sin x \cos x + 2\cos^2 x = 0$ is
   A) 1 B) 2 C) 3 D) 4.

7. The minimum value of the function $f(x) = x^x$, $x \in (0, \infty)$ is
   A) $\left(\frac{1}{10}\right)^{\frac{1}{10}}$ B) $10^{\frac{1}{10}}$ C) $\frac{1}{e}$ D) $\left(\frac{1}{e}\right)^{\frac{1}{e}}$

8. Let f be a twice differentiable function on $\mathbb{R}$. Also $f''(x) > 0$ for all $x \in \mathbb{R}$. Which of the following statements is true?
   A) $f(x) = 0$ has exactly two solutions on $\mathbb{R}$.
   B) $f(x) = 0$ has a positive solution if $f(0) = 0$ and $f'(0) = 0$.
   C) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) > 0$.
   D) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) < 0$.

9. If $x^2+x+1=0$, then the value of $\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\ldots+\left(x^{27}+\frac{1}{x^{27}}\right)^2$ is

   A) 27 B) 54 C) 0 D) -27.

10. Let M= $\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$. Then $M^{2017}=$

    A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ B) $\begin{pmatrix} -2^{2017} & -1 \\ 3^{2017} & 1 \end{pmatrix}$ C) $\begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$ D) $\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let a,b,c be real numbers such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $abc=1$. Prove that at least one of a,b,c is 1.

2. Let $c_1,c_2,\ldots,c_9$ be the zeros of the polynomial $z^9-6z^7+12z^6+18z^4-24z^3+30z^2-z+2017$. If $S(z)=\sum_{k=1}^{9}|z-c_k|^2$, then prove that $S(z)$ is constant on the circle $|z|=100$.

3. Let f be a monic polynomial with real coefficients. Let $\lim_{x\to\infty} f''(x)=\lim_{x\to\infty} f\left(\frac{1}{x}\right)$ and $f(x)\geq f(1)$ for all $x\in\mathbb{R}$. Find f.

4. Call a set of integers *non − isolated* if for every $a\in A$ at least one of the numbers $a-1$ and $a+1$ also belongs to A. Prove that the number of 5−element *non − isolated* subsets of $\{1,2,\ldots,n\}$ is $(n-4)^2$.

5. Find all positive integers n for which a permutation $a_1,a_2,\ldots,a_n$ of $\{1,2,\ldots,n\}$ can be found such that $0,a_1,a_1+a_2,a_1+a_2+a_3,\ldots,a_1+a_2+\ldots+a_n$ leave distinct remainders modulo $n+1$.

Part III

1. Do there exist 100 lines in the plane, no three concurrent such that they intersect exactly in 2017 points? [12]

2. On the parabola $y=x^2$, a sequence of points $P_n(x_n,y_n)$ is selected recursively where the points P_1,P_2 are arbitrarily selected distinct points. Having selected P_n, tangents drawn at P_{n-1} and P_n meet at say Q_n. Suppose P_{n+1} is the point of intersection of $y=x^2$ and the line passing through Q_n parallel to Y-axis. Under what conditions on P_1,P_2

   (a) both the sequences $\{x_n\}$ and $\{y_n\}$ converge?

   (b) $\{x_n\}$ and $\{y_n\}$ both converge to 0? [13]

3. (a) Show that there does not exist a 3-digit number A such that 10^3A+A is a perfect square.

   (b) Show that there exists an n-digit ($n>3$) number A such that 10^nA+A is a perfect square. [12]

4. For $n\times n$ matrices A,B, let $C=AB-BA$. If C commutes with both A and B, then

   (a) Show that $AB^k-B^kA=kB^{k-1}C$ for every positive integer k.

   (b) Show that there exists a positive integer m such that $C^m=0$. [13]

MADHAVA MATHEMATICS COMPETITION, January 8, 2017
Solutions and scheme of marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The number $\sqrt{2}e^{i\pi}$ is:
   A) a rational number.
   B) an irrational number.
   C) a purely imaginary number.
   D) a complex number of the type $a + ib$ where a, b are non-zero real numbers.
   Answer: B
   The number is $-\sqrt{2}$ using the relation $e^{i\pi} = -1$.

2. Let $P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The rank of P^4 is: A) 1 B) 2 C) 3 D) 4.
   Answer: D
   Det $P \neq 0, \Rightarrow$ Det $P^4 \neq 0$. Thus rank of P^4 is 4.

3. Let $y_1(x)$ and $y_2(x)$ be the solutions of the differentiable equation $\dfrac{dy}{dx} = y + 17$ with initial conditions $y_1(0) = 0, y_2(0) = 1$. Which of the following statements is true?
   A) y_1 and y_2 will never intersect.
   B) y_1 and y_2 will intersect at $x = e$.
   C) y_1 and y_2 will intersect at $x = 17$.
   D) y_1 and y_2 will intersect at $x = 1$.
   Answer: A
   Solving the differentiable equation we get $y_1 = -17 + 17e^x$ and $y_2 = -17 + 18e^x$. The two curves never intersect.

4. Suppose f and g are differentiable functions and $h(x) = f(x)g(x)$. Let $h(1) = 24$, $g(1) = 6$, $f'(1) = -2$, $h'(1) = 20$. Then the value of $g'(1)$ is
   A) 8 B) 4 C) 2 D) 16.
   Answer: A
   $h(x) = f(x)g(x)$. Thus we get $h^{'}(x) = f^{'}(x)g(x) + f(x)g^{'}(x)$
   $h^{'}(1) = f^{'}(1)g(1) + f(1)g^{'}(1)$. Therefore $20 = (-2)(6) + f(1)g^{'}(1)$.
   $f(1)g^{'}(1) = 32$. Now $h(1) = f(1)g(1)$. Therefore $24 = 6f(1)$.
   Thus $f(1) = 4$ and $g^{'}(1) = 8$.

5. In how many regions is the plane divided when the following equations are graphed, not considering the axes? $y = x^2,\ y = 2^x$
   A) 3 B) 4 C) 5 D) 6.
   Answer: D
   Plot graph of the two functions $y = x^2$ and $y = 2^x$.

6. For $0 \leq x < 2\pi$, the number of solutions of the equation $\sin^2 x + 3\sin x \cos x + 2\cos^2 x = 0$ is
   A) 1 B) 2 C) 3 D) 4.
   Answer: D
   Note $\cos^2(x) \neq 0$. Dividing by $\cos^2(x)$ we get $\tan^2(x) + 3\tan x + 2 = 0$. Thus $\tan x = -1$ or -2. Since $\tan x$ has period π and range of $\tan x$ is $(-\infty, \infty)$, the number of solutions of the given equation in the interval $0 \leq x < 2\pi$ is equal to 4.

7. The minimum value of the function $f(x) = x^x,\ x \in (0, \infty)$ is

   A) $\left(\frac{1}{10}\right)^{\frac{1}{10}}$ B) $10^{\frac{1}{10}}$ C) $\frac{1}{e}$ D) $\left(\frac{1}{e}\right)^{\frac{1}{e}}$.

   Answer: D

   $y = x^x$. Taking log on both sides $\ln y = x \ln x$.

   $\frac{1}{y}\frac{dy}{dx} = \ln x + 1$. Solving $\frac{dy}{dx} = 0 \Rightarrow \ln_e x = -1$. Thus minimum value is obtained at $x = 1/e$ and is $\left(\frac{1}{e}\right)^{\frac{1}{e}}$. Note that $\frac{d^2y}{dx^2}(\frac{1}{e}) > 0$.

8. Let f be a twice differentiable function on $\mathbb{R}$. Also $f''(x) > 0$ for all $x \in \mathbb{R}$. Which of the following statements is true?

   A) $f(x) = 0$ has exactly two solutions on $\mathbb{R}$.

   B) $f(x) = 0$ has a positive solution if $f(0) = 0$ and $f'(0) = 0$.

   C) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) > 0$.

   D) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) < 0$.

   Answer: C

   $f''(x) > 0 \Rightarrow f'(x)$ is increasing. Also $f'(0) > 0 \Rightarrow f'(x) > 0$ if $x > 0$. $\Rightarrow f(x) = 0$ has no positive solution.

9. If $x^2 + x + 1 = 0$, then the value of $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \ldots + \left(x^{27} + \frac{1}{x^{27}}\right)^2$ is

   A) 27 B) 54 C) 0 D) -27.

   Answer: B

   $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \cdots + \left(x^{27} + \frac{1}{x^{27}}\right)^2$

   $= \left(\frac{x^2+1}{x}\right)^2 + \left(\frac{x^4+1}{x^2}\right)^2 + \cdots + \left(\frac{x^{54}+1}{x^{27}}\right)^2$

   $= 9((-1)^2 + (-1)^2 + 2^2) = 54$.

10. Let M= $\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$. Then $M^{2017} =$

    A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ B) $\begin{pmatrix} -2^{2017} & -1 \\ 3^{2017} & 1 \end{pmatrix}$ C) $\begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$ D) $\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$.

    Answer: D

    Note $M^3 = I$. Thus $M^{2016} = I$ and $M^{2017} = M$.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let a, b, c be real numbers such that $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $abc = 1$. Prove that at least one of a, b, c is 1.

   Solution: Let $\lambda = a + b + c$. Then $\lambda = a + b + c = \frac{bc + ac + ab}{abc} = bc + ac + ab$. **2 marks**

   The numbers a, b, c are roots of the polynomial $x^3 - \lambda x^2 + \lambda x - 1$. Observe that $x = 1$ is one of the roots of this polynomial. **4 marks**

2. Let $c_1, c_2, \ldots, c_9$ be the zeros of the polynomial $z^9 - 6z^7 + 12z^6 + 18z^4 - 24z^3 + 30z^2 - z + 2017$. If $S(z) = \sum_{k=1}^{9} |z - c_k|^2$, then prove that $S(z)$ is constant on the circle $|z| = 100$.

   Solution: Observe that $\sum_{k=1}^{9} c_k = 0$. **2 marks**

$S(z) = \sum_{k=1}^{9} |z - c_k|^2 = \sum_{k=1}^{9} (z - c_k)\overline{(z - c_k)} = \sum_{k=1}^{9} (z\overline{z} - z\overline{c_k} - c_k\overline{z} + c_k\overline{c_k}) = \sum_{k=1}^{9} |z|^2 + \sum_{k=1}^{9} |c_k|^2 - z\sum_{k=1}^{9} \overline{c_k} - \overline{z}\sum_{k=1}^{9} c_k = \sum_{k=1}^{9} |z|^2 + \sum_{k=1}^{9} |c_k|^2 = \sum_{k=1}^{9} (100)^2 + \sum_{k=1}^{9} |c_k|^2 = \text{constant}.$ **4 marks**

3. Let f be a monic polynomial with real coefficients. Let $\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} f\left(\frac{1}{x}\right)$ and $f(x) \geq f(1)$ for all $x \in \mathbb{R}$. Find f.
Solution: Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Then $f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$ and $f''(x) = 2a_2 + 6a_3x + \cdots + n(n-1)a_nx^{n-2}$. Also $f(\frac{1}{x}) = a_0 + a_1(\frac{1}{x}) + \cdots + a_n(\frac{1}{x})^n$. Now $\lim_{x\to\infty} f\left(\frac{1}{x}\right) = a_0$. Therefore $\lim_{x\to\infty} f''(x) = a_0$. Hence $f''(x) = 2a_2$ and $a_3 = a_4 = \cdots = a_n = 0$. **3 marks**
Since f is monic, $a_2 = 1$. Now $\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} f\left(\frac{1}{x}\right)$ implies $a_0 = 2a_2 = 2$. Also $f(x) \geq f(1)$ for all $x \in \mathbb{R}$ implies f has minimum at $x = 1$. Therefore $f'(1) = 0$. Hence $a_1 + 2a_2 = 0$. Therefore $a_1 = -2$. Hence $f(x) = x^2 - 2x + 2$. **3 marks**

4. Call a set of integers $non - isolated$ if for every $a \in A$ at least one of the numbers $a-1$ and $a+1$ also belongs to A. Prove that the number of $5-$element $non - isolated$ subsets of $\{1, 2, \ldots, n\}$ is $(n-4)^2$.
Solution: Let $\{a_1, a_2, a_3, a_4, a_5\}$ be 5-element $non - isolated$ subset of $\{1, 2, \ldots, n\}$ such that $a_1 < a_2 < a_3 < a_4 < a_5$. Then $a_2 = a_1 + 1, a_4 = a_5 - 1$. Further $a_3 = a_2 + 1$ or $a_3 = a_4 - 1$. Clearly $1 \leq a_1 \leq n-4$. For each choice of a_1, a_5 has $(n-3) - a_1$ choices. For a_3, there are 2 choices. So total number of such sets is $2[(n-4) + (n-5) + \cdots + 1] = 2\frac{(n-4)(n-3)}{2} = (n-4)(n-3)$. **4 marks**
But $a_3 = a_2 + 1$ as well as $a_3 = a_4 - 1$ gets counted twice. So total number of such sets is $(n-4)(n-3) - (n-4) = (n-4)(n-4) = (n-4)^2$. **2**

5. Find all positive integers n for which a permutation $a_1, a_2, \ldots, a_n$ of $\{1, 2, \ldots, n\}$ can be found such that $0, a_1, a_1+a_2, a_1+a_2+a_3, \ldots, a_1+a_2+\ldots+a_n$ leave distinct remainders modulo $n+1$.
Solution: Let $a_1, a_2, \ldots, a_n$ be a permutation of $\{1, 2, \ldots, n\}$ such that $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \ldots + a_n$ leave distinct remainders modulo $n+1$. If n is even, then $a_1 + a_2 + \cdots + a_n = \frac{n(n+1)}{2} \equiv 0(\mod n+1)$. So remainders modulo $n+1$ can not be distinct. **3 marks**
Let n be odd. Then choose $a_1 = 1, a_2 = n-1, a_3 = 3, a_4 = n-3$ and so on. Then $a_1 \equiv 1(\mod n+1), a_1 + a_2 \equiv n(\mod n+1), a_1 + a_2 + a_3 = n + 3 \equiv 2(\mod n+1), a_1 + a_2 + a_3 + a_4 = 2n \equiv n - 1(\mod n+1), \cdots$. This gives required permutation. **3 marks**

Part III

1. Do there exist 100 lines in the plane, no three concurrent such that they intersect exactly in 2017 points? [12]
Solution: Consider k sets of parallel lines having respectively $m_1, m_2, \cdots, m_k$ lines in each set. Then we need to solve the equations $m_1 + m_2 + \cdots + m_k = 100$ and $\sum_{i<j} m_i m_j = 2017$. **4 marks**
$\left(\sum m_i\right)^2 = \sum m_i^2 + 2\sum_{i<j} m_i m_j$

Therefore $\sum m_i^2 = 10,000 - 2(2017) = 5966$. Thus we need to find 100 numbers satisfying $\sum m_i = 100$ and $\sum m_i^2 = 5966$. **3 marks**
We can get any one of the following solutions:

(a) $75, 18, 3, 2, 2$

(b) $77, 3, 2, 2, 2, 2, 1$(12 times)

(c) $77, 4, 2, 1$(17 times) **5 marks**

2. On the parabola $y = x^2$, a sequence of points $P_n(x_n, y_n)$ is selected recursively where the points P_1, P_2 are arbitrarily selected distinct points. Having selected P_n, tangents drawn at P_{n-1} and P_n meet at say Q_n. Suppose P_{n+1} is the point of intersection of $y = x^2$ and the line passing through Q_n parallel to Y-axis. Under what conditions on P_1, P_2

(a) both the sequences $\{x_n\}$ and $\{y_n\}$ converge?

(b) $\{x_n\}$ and $\{y_n\}$ both converge to 0? [13]

Solution:

(a) Since $y_n = x_n^2$, it is enough to discuss the convergence of $\{x_n\}$. Tangents at x_n, x_{n-1} are given by $y = x_n^2 + 2x_n(x - x_n) = 2x_n x - x_n^2$ and $y = 2x_{n-1}x - x_{n-1}^2$.
Solving we get, $x_{n+1} = \dfrac{x_{n-1} + x_n}{2}$. Therefore $P_{n+1}\left(\dfrac{x_{n-1} + x_n}{2}, (\dfrac{x_{n-1} + x_n}{2})^2\right)$. **4 marks**
Now all x_n are within the interval $[x_1, x_2]$ and all are distinct. Hence $\{x_n\}$ converges. **3 marks**

(b) We first consider a special case where $x_1 = 0, x_2 = 1$. Then the sequence $\{x_n\}$ is $0, 1, \dfrac{1}{2}, \dfrac{3}{4}, \dfrac{1}{2} + \dfrac{1}{8}, \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{16}, \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{32}, \cdots$. Its sub sequence is $\dfrac{1}{2}, \dfrac{1}{2} + \dfrac{1}{8}, \dfrac{1}{2} + \dfrac{1}{8} + \dfrac{1}{32}, \cdots$ which are partial sums of the geometric series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}}$ which converges and sum is given by $\dfrac{2}{3}$. Thus in this case x_n convereges to $\frac{2}{3}$.
Now in general, for any $x_1 < x_2$, we define $z_{n-1} = \dfrac{x_n - x_1}{x_2 - x_1}$. Note that $z_0 = 0$ and $z_1 = 1$ and the sequence z_n satisfies the relation $z_{n+1} = \dfrac{z_{n-1} + z_n}{2}$. Thus by the special case above, z_n converges to $\frac{2}{3}$. Now observe that the limit of x_n is the real number x which divides the interval $[x_1, x_2]$ in the ratio $2 : 1$. For $x = 0$, we need to take $x_1 \neq 0$ and $x_2 = -\dfrac{1}{2}x_1$. **6 marks**

3. (a) Show that there does not exist a 3-digit number A such that $10^3A + A$ is a perfect square.

(b) Show that there exists an n-digit ($n > 3$) number A such that $10^nA + A$ is a perfect square. [12]

Solution:

(a) If $10^3A + A = A(1001)$ is a perfect square, then $7|A, 11|A$ and $13|A$. Therefore $A \geq 1001$. That is A has bigger than or equal to 4 digits. **3 marks**

(b) For some n if $10^nA + A = A(10^n + 1)$ is a perfect square, then $10^n + 1$ must be divisible by a square bigger than 1. Because if no perfect square divides $10^n + 1$, then all prime divisors of $10^n + 1$ must appear in factorization of A. This makes $A \geq 10^n + 1$, but $A < 10^n + 1$. Therefore $10^n + 1$ has a square factor. **4 marks**

Now observe that $10^{11}+1 = (11)(10^{10}-10^9+10^8-\cdots-10+1) = (11)(9090909091)$ and thus 121 divides $10^{11}+1$. **2 marks**

Thus we can choose A as $\dfrac{10^{11}+1}{121} \times 9$ so that $10^{10} \leq A < 10^{11}$ and $(10^n+1)A$ is a perfect square. **3 marks**

4. For $n \times n$ matrices A, B, let $C = AB - BA$. If C commutes with both A and B, then

   (a) Show that $AB^k - B^kA = kB^{k-1}C$ for every positive integer k.

   (b) Show that there exists a positive integer m such that $C^m = 0$. [13]

   Solution:

   (a) The proof is by induction. The result is true for $k = 1, 2$. $AB - BA = C$ and $AB^2 - B^2A = (AB - BA)B + B(AB - BA) = CB + BC = 2BC$. **1 marks**
   Assume that the result is true upto $k-1$. $AB^k - B^kA = (AB - BA)B^{k-1} + B(AB^{k-1} - B^{k-1}A) = CB^{k-1} + B(k-1)B^{k-2}C = kB^{k-1}C$. **3 marks**

   (b) Hence for any polynomial $q(x)$, $Aq(B) - q(B)A = q'(B)C$, where q' is a derivative of q. In particular, let $p(x)$ be the characteristic polynomial of B. Now by Caley-Hamilton Theorem,
   $0 = Ap(B) - p(B)A = p'(B)C$. This proves $p'(B)C = 0$. **4 marks**
   $0 = Ap'(B)C - p'(B)AC = (Ap'(B) - p'(B)A)C = p''(B)C^2$. This proves $p''(B)C^2 = 0$. Inductively, we have $p^{(k)}(B)C^k = 0$. Therefore for $k = n$, we have $n!C^n = 0$. Hence $C^n = 0$. **5 marks**

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 07/01/2018 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Suppose $\lim_{x\to -2} \dfrac{bx^2+15x+15+b}{x^2+x-2} = L$, then the values of b and L are respectively
   A) $3, -1$ B) $-3, 1$ C) $3, 1$ D) $-3, -1$.

2. Let f be a function defined on the set of positive integers by $f(1) = 1$, $f(2n) = 2f(n), f(2n+1) = 4f(n)$. The number of solutions to $f(n) = 8$ equals
   A) 4 B) 3 C) 2 D) 1.

3. Let g be a continuous function which is not differentiable at 0 and let $g(0) = 8$. If $f(x) = xg(x)$, then $f'(0) =$
   A) 0 B) 4 C) 2 D) 8.

4. If the circles $x^2+y^2 = 1$ and $(x-a)^2+(y-b)^2 = 1$ have exactly one point in common, then the point (a, b) lies on
   A) $x^2+y^2 = 1$ B) $(x-a)^2+(y-b)^2 = 1$ C) $x^2+y^2 = 2$ D) $x^2+y^2 = 4$.

5. If x and y are **non-zero** real numbers, then x^2+xy+y^2
   A) is always negative B) takes the value zero for some x, y
   C) is always positive D) takes both positive and negative values.

6. The set of all real numbers x such that $\Big||3-x|-|x+2|\Big| = 5$ is
   A) $[3, \infty)$ B) $(-\infty, -2] \cup [3, \infty)$ C) $(-\infty, -2]$ D) $(-\infty, -3] \cup [2, \infty)$.

7. Let $f(x) = \sin x + \cos x$. The infimum of $f(x)$ over the interval $[0, \pi/4]$ is
   A) 0 B) 1 C) $\sqrt{2}$ D) $1/\sqrt{2}$.

8. Let A be the set of points where the function $f(x) = \cos|x-5| + |x+10|^3$ is not differentiable. Then
   A) $A = \{5\}$ B) $A = \{5, -10\}$ C) $A = \{-10\}$ D) $A = \phi$.

9. How many factors of $2^5 3^6 5^2$ are perfect squares?
   A) 24 B) 20 C) 30 D) 36.

10. The number of distinct real roots of the equation $2x^5 + 8x - 7 = 0$ is
    A) 1 B) 2 C) 3 D) 5.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $A_1 = \{1\}, A_{n+1} = \{3k, 3k+1 : \ k \in A_n\}$ for all $n \geq 1$ and $A = \bigcup_{n=1}^{\infty} A_n$. Can 2017 be written as the sum of two elements of A? Is this representation unique?

2. Prove that there do not exist distinct real numbers x, y, u, v satisfying $x^2 + y^2 = u^2 + v^2$ and
$x^3 + y^3 = u^3 + v^3$ simultaneously.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that $\int_0^1 f(x)x^2 dx = \frac{1}{3}f(c)$ for some $c \in [0, 1]$.

4. Suppose $f(x)$ is a polynomial with real coefficients and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Let $g = f + f' + f'' + \cdots$. Prove that $g(x) \geq 0$ for all $x \in \mathbb{R}$.

5. Let S be the set of all points $(a, b) \in \mathbb{R}^2$ for which the curves $y = x^2 + 2bx + 1$ and $y = 2a(x + b)$ do not intersect. Determine the area of S.

Part III

1. Two real numbers x and y are chosen in $(0, 1)$ randomly.

   (a) Show that the probability that 2 is the closest integer to $\frac{y}{x}$ is $2/15$.

   (b) If P_1 is the probability that the integer closest to $\frac{y}{x}$ is odd and if P_2 is the probability that the integer closest to $\frac{y}{x}$ is even then show that $P_1 > P_2$. [12]

2. (a) If α is a root of the polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ with real coefficients, $a_n \neq 0$, then prove that $|\alpha| \leq 1 + \max_{0 \leq k \leq n-1} |\frac{a_k}{a_n}|$.

   (b) Let $a_0 + 10a_1 + \cdots + 10^n a_n$ be the decimal representation of a prime number such that $a_n \geq 2, n > 1$. Prove that the polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ cannot be written as a product of two non-constant polynomials with integer coefficients. [12]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + \tan^{-1} x$.

   (a) Show that f is bijection.

   (b) Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} f(x) & \text{if } \ x \geq c, \\ f^{-1}(x) & \text{if } \ x < c. \end{cases}$$

   Find all real numbers c such that h is continuous at c.

   (c) Find all real numbers c such that h is differentiable at c. [13]

4. Let A, B be $n \times n$ $(n \geq 2)$ nonsingular matrices with real entries.

   (a) If $A^{-1} + B^{-1} = (A + B)^{-1}$, then prove that $\det A = \det B$.

   (b) Find examples of matrices A, B satisfying $A^{-1} + B^{-1} = (A + B)^{-1}$.

   (c) Find examples of matrices A, B with complex entries such that $A^{-1} + B^{-1} = (A + B)^{-1}$, but $\det A \neq \det B$. [13]

MADHAVA MATHEMATICS COMPETITION, January 7, 2018
Solutions and scheme of marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Suppose $\lim_{x\to -2} \dfrac{bx^2+15x+15+b}{x^2+x-2} = L$, then the values of b and L are respectively
A) $3, -1$ B) $-3, 1$ C) $3, 1$ D) $-3, -1$.
Answer: A
Put $x = -2$ in $bx^2+15x+15+b = 0$. We get $b = 3$. By L Hospital rule, the value of the limit is -1.

2. Let f be a function defined on the set of positive integers by $f(1) = 1$, $f(2n) = 2f(n), f(2n+1) = 4f(n)$. The number of solutions to $f(n) = 8$ equals
A) 4 B) 3 C) 2 D) 1.
Answer: B
Case 1: n is even i.e. $n = 2k$. Then $f(2k) = 2f(k) = 8$. Therefore $f(k) = 4$. If k is even, then $k = 2m$ and $f(2m) = 2f(m) = 4$. Therefore $f(m) = 2$. Therefore $m = 2, k = 4$ and $n = 8$. If k is odd, then $k = 2m+1$ and $f(2m+1) = 4f(m) = 4$. Therefore $f(m) = 1$. Therefore $m = 1, k = 3$ and $n = 6$.
Case 2: n is odd i.e. $n = 2k+1$. Then $f(2k+1) = 4f(k) = 8$. Therefore $f(k) = 2$. Then $k = 2m$ and $f(2m) = 2f(m) = 2$. Therefore $f(m) = 1$. Hence $m = 1, k = 2$ and $n = 5$. Therefore there are 3 solutions to the equation $f(n) = 8$.

3. Let g be a continuous function which is not differentiable at 0 and let $g(0) = 8$. If $f(x) = xg(x)$, then $f'(0) =$
A) 0 B) 4 C) 2 D) 8.
Answer: D
$f'(0) = \lim_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim_{h\to 0} \dfrac{hg(h)-f(0)}{h} = \lim_{h\to 0} g(h) = g(0) = 8.$

4. If the circles $x^2+y^2 = 1$ and $(x-a)^2+(y-b)^2 = 1$ have exactly one point in common, then the point (a, b) lies on
A) $x^2+y^2 = 1$ B) $(x-a)^2+(y-b)^2 = 1$ C) $x^2+y^2 = 2$ D) $x^2+y^2 = 4$.
Answer: D
Note that the distance between (a, b) and $(0, 0)$ is 2.

5. If x and y are **non-zero** real numbers, then x^2+xy+y^2
A) is always negative B) takes the value zero for some x, y
C) is always positive D) takes both positive and negative values.
Answer: C
Observe that x^2+xy+y^2 is positive when both x, y are positive or both x, y are negative. Note that $x^2+xy+y^2 = (x+y)^2-xy$, is positive if one of x, y is positive and other is negative. Hence the value is always positive.

6. The set of all real numbers x such that $\Big||3-x|-|x+2|\Big| = 5$ is
A) $[3, \infty)$ B) $(-\infty, -2] \cup [3, \infty)$ C) $(-\infty, -2]$ D) $(-\infty, -3] \cup [2, \infty)$.
Answer: B
Consider three cases: $(-\infty, -2], (-2, 3), [3, \infty)$. Observe that all real numbers $x \in (-\infty, -2] \cup [3, \infty)$ satisfy the given equation.

7. Let $f(x) = \sin x + \cos x$. The infimum of $f(x)$ over the interval $[0, \pi/4]$ is
A) 0 B) 1 C) $\sqrt{2}$ D) $1/\sqrt{2}$.

Answer: B
Observe that $f'(x) = \cos x - \sin x > 0$ on $[0, \pi/4]$. Therefore f is strictly increasing. Hence the minimum value of $f(x)$ is at 0, which is 1.

8. Let A be the set of points where the function $f(x) = \cos|x-5| + |x+10|^3$ is not differentiable. Then
A) $A = \{5\}$ B) $A = \{5, -10\}$ C) $A = \{-10\}$ D) $A = \phi$.
Answer: D
Observe that f is differentiable everywhere.

9. How many factors of $2^5 3^6 5^2$ are perfect squares?
A) 24 B) 20 C) 30 D) 36.
Answer: A
A factor of $2^5 3^6 5^2$ which is a perfect square is of the form $2^\alpha 3^\beta 5^\gamma$, where the powers are even integers. Therefore α has 3 choices $0, 2, 4$ and β has 4 choices $0, 2, 4, 6$ and γ has 2 choices $0, 2$. Hence total required factors are 24.

10. The number of distinct real roots of the equation $2x^5 + 8x - 7 = 0$ is
A) 1 B) 2 C) 3 D) 5.
Answer: A
For $f(x) = 2x^5 + 8x - 7$, $f'(x) = 10x^4 + 8 > 0$. Therefore f is strictly increasing. Note that $f(0) < 0$ and $f(1) > 0$. Hence by intermediate value property, there is a root between 0 and 1. Thus the equation $f(x) = 0$ has only one root.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $A_1 = \{1\}, A_{n+1} = \{3k, 3k+1 : \ k \in A_n\}$ for all $n \geq 1$ and $A = \bigcup_{n=1}^{\infty} A_n$. Can 2017 be written as the sum of two elements of A? Is this representation unique?
Solution: Observe that $A = \{1\} \cup \{3, 4\} \cup \{9, 10, 12, 13\} \cup \cdots$. The ternary expansion of numbers in A_{n+1} is obtained by adding 0 or 1 at the end of ternary expansion of numbers in A_n. In terms of ternary system, $A = \{1\} \cup \{10, 11\} \cup \{100, 101, 110, 111\} \cup \cdots$. Observe that a number is in A if and only if its ternary expansion consists of only 0s and 1s. [3]
$2017 = (2202201)_3 = (1101101)_3 + (1101100)_3$. [1]
Further, a number N can be uniquely written as sum of two elements of A if and only if the ternary expansion of N has at most one 1. Since the ternary expansion of 2017 contains only one 1, the representation is unique. [2]

2. Prove that there do not exist distinct real numbers x, y, u, v satisfying $x^2 + y^2 = u^2 + v^2$ and $x^3 + y^3 = u^3 + v^3$ simultaneously.
Solution: Let $x^3 = x_1, y^3 = y_1, u^3 = u_1, v^3 = v_1$ and
$x_1 + y_1 = u_1 + v_1$, $x_1^{2/3} + y_1^{2/3} = u_1^{2/3} + v_1^{2/3}$. Without loss of generality, the given relations allow us to choose x_1, y_1, u_1, v_1 such that $x_1 > u_1 > v_1 > y_1$. [2]
Applying LMVT to the function $f(x) = x^{2/3}$ on the intervals $[u_1, x_1]$ and $[y_1, v_1]$, we get $x_1^{2/3} - u_1^{2/3} = (2/3)t_1^{-1/3}(x_1 - u_1)$ for some $t_1 \in (u_1, x_1)$ and
$v_1^{2/3} - y_1^{2/3} = (2/3)t_2^{-1/3}(v_1 - y_1)$ for some $t_2 \in (y_1, v_1)$.
This implies $t_1 = t_2$, which is a contradiction because t_1 and t_2 belong to disjoint intervals. [4]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that $\displaystyle\int_0^1 f(x)x^2 dx = \frac{1}{3}f(c)$ for some $c \in [0, 1]$.
Solution: Since f is continuous on $[0, 1]$, it is bounded there and attains its bounds,

say m and M. [2]

Suppose $m \le f(x) \le M$ for all $x \in [0,1]$.

Multiplying all sides by x^2, we get $mx^2 \le f(x)x^2 \le Mx^2$ for all $x \in [0,1]$.

Integrating all sides between 0 and 1, we get $m(1/3) \le \int_0^1 f(x)x^2dx \le M(1/3)$.

This implies $m \le 3\int_0^1 f(x)x^2dx \le M$. [3]

By intermediate value property, there exists $c \in [0,1]$ such that $f(c) = 3\int_0^1 f(x)x^2dx$. [1]

4. Suppose $f(x)$ is a polynomial with real coefficients and $f(x) \ge 0$ for all $x \in \mathbb{R}$. Let $g = f + f' + f'' + \cdots$. Prove that $g(x) \ge 0$ for all $x \in \mathbb{R}$.

Solution: It is given that $f(x) \ge 0$ for all $x \in \mathbb{R}$. Observe that the degree of the polynomials $f(x)$ and $g(x)$ is same and it is even and they have positive leading coefficients. Note that $g' = g - f$. As $x \to \infty,\ g(x) \to \infty$ and as $x \to -\infty,\ g(x) \to \infty$. Therefore it can be seen that minimum of $g(x)$ exists. [3]

Let $g(x) < 0$ at some point. If g has minimum at x_0, then $g(x_0) < 0$. Observe that $g'(x_0) = g(x_0) - f(x_0) < 0$. But, $g'(x_0) = 0$, which is contradiction. Hence $g(x) \ge 0$ for all $x \in \mathbb{R}$. [3]

5. Let S be the set of all points $(a,b) \in \mathbb{R}^2$ for which the curves $y = x^2 + 2bx + 1$ and $y = 2a(x+b)$ do not intersect. Determine the area of S.

Solution: Equating $x^2 + 2bx + 1 = 2a(x+b)$, we get $x^2 + 2(b-a)x + (1-2ab) = 0$. The discriminant is $4(b-a)^2 - 4(1-2ab)$. The curves $y = x^2 + 2bx + 1$ and $y = 2a(x+b)$ do not intersect, if discriminant is negative. [4]

Therefore $4(b-a)^2 - 4(1-2ab) < 0$. This implies $a^2 + b^2 < 1$. Hence S is an open unit disc and area of S is π. [2]

Part III

1. Two real numbers x and y are chosen in $(0,1)$ randomly.

   (a) Show that the probability that 2 is the closest integer to $\dfrac{y}{x}$ is $2/15$.

   (b) If P_1 is the probability that the integer closest to $\dfrac{y}{x}$ is odd and if P_2 is the probability that the integer closest to $\dfrac{y}{x}$ is even then show that $P_1 > P_2$. [12]

Solution: In order that closest integer to $\dfrac{y}{x}$ is 2, we must have $3/2 < y/x < 5/2$ i.e. (x,y) lies in the region R, where R is the region bounded by the lines $y = (3/2)x, y = (5/2)x$ and $y = 1$. The probability that $\dfrac{y}{x}$ is closest to 2 is the area of the region R which is $= \dfrac{1}{2}\left(\dfrac{2}{3} - \dfrac{2}{5}\right) = \dfrac{1}{3} - \dfrac{1}{5} = \dfrac{2}{15}$. [4]

Similarly probability that $\dfrac{y}{x}$ is closest to 4 is $= \dfrac{1}{2}\left(\dfrac{2}{7} - \dfrac{2}{9}\right) = \dfrac{1}{7} - \dfrac{1}{9}$.

Probability that $\dfrac{y}{x}$ is closest to 0 is $1/4$. It is area above $y = x$ in the unit square. Therefore the probability that the integer closest to $\dfrac{y}{x}$ is even i.e.

$P_2 = \dfrac{1}{4} + \left(\dfrac{1}{3} - \dfrac{1}{5}\right) + \left(\dfrac{1}{7} - \dfrac{1}{9}\right) + \cdots$ [2]

Similar calculations show that the probability that the integer closest to $\dfrac{y}{x}$ is odd i.e.

$P_1 = \dfrac{5}{12} + \left(\dfrac{1}{5} - \dfrac{1}{7}\right) + \left(\dfrac{1}{9} - \dfrac{1}{11}\right) + \cdots$ [2]

We see that $\frac{5}{12} > \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{5}\right)$.
Also $\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right) > \left(\frac{1}{2k+3} - \frac{1}{2k+5}\right)$ for all $k \geq 2$.
Adding all the above inequalities, we get that $P_1 > P_2$. [4]

2. (a) If α is a root of the polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ with real coefficients, $a_n \neq 0$, then prove that $|\alpha| \leq 1 + \max\limits_{0 \leq k \leq n-1} \left|\frac{a_k}{a_n}\right|$.

   (b) Let $a_0 + 10a_1 + \cdots + 10^n a_n$ be the decimal representation of a prime number such that $a_n \geq 2, n > 1$. Prove that the polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ cannot be written as a product of two non-constant polynomials with integer coefficients. [12]

Solution:

(a) Let α be a root of the polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $-a_n\alpha^n = a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$. Let $M = \max\limits_{0 \leq k \leq n-1} \left|\frac{a_k}{a_n}\right|$. If $|\alpha| = 1$, then result holds. Let $|\alpha| \neq 1$. We have
$|\alpha|^n \leq \left|\frac{a_0}{a_n}\right| + \left|\frac{a_1}{a_n}\right| |\alpha| + \cdots + \left|\frac{a_{n-1}}{a_n}\right| |\alpha|^{n-1} \leq M(1 + |\alpha| + \cdots + |\alpha|^{n-1}) \leq M\frac{|\alpha|^n - 1}{|\alpha| - 1}$.
Then $|\alpha|^n(|\alpha| - 1) \leq M|\alpha|^n - M$.
$|\alpha|^n(|\alpha| - 1 - M) \leq -M$. This implies $|\alpha| - 1 - M < 0$. Therefore $|\alpha| < M + 1$. [6]

(b) Note that $a_n \geq 2$ and each $|a_k| \leq 9$. Then $\left|\frac{a_k}{a_n}\right| \leq \frac{9}{2}$ for all k. By part (a), $|\alpha| \leq 1 + \frac{9}{2} = \frac{11}{2}$.
Let if possible $p(x) = q(x)r(x)$. Then $p(10) = q(10)r(10)$. But $p(10)$ is prime, therefore $q(10)$ or $r(10)$ is 1. Let $q(10) = 1$. Let $q(x) = c(x - x_1) \cdots (x - x_t)$. Then $q(10) = 1 = c(10 - x_1) \cdots (10 - x_t)$. But each $|10 - x_i| \geq 10 - |x_i| \geq 10 - 11/2 = 9/2$.
Then $1 \geq |c|\frac{9}{2}\frac{9}{2} \cdots \frac{9}{2}$, which is a contradiction. [6]

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + \tan^{-1} x$.

   (a) Show that f is bijection.

   (b) Define $h : \mathbb{R} \to \mathbb{R}$ by
$$h(x) = \begin{cases} f(x) & \text{if } x \geq c, \\ f^{-1}(x) & \text{if } x < c. \end{cases}$$
   Find all real numbers c such that h is continuous at c.

   (c) Find all real numbers c such that h is differentiable at c. [13]

Solution:

(a) We first show that f is injective. Since f is differentiable, consider f'. We see that
$$f'(x) = 1 + \frac{1}{1 + x^2}, \quad \forall x \in \mathbb{R}$$
From the above expression, we see that $f'(x) > 0$, $\forall x \in \mathbb{R}$. This implies that the function f is an increasing function and hence, injective. To show that f is surjective, we notice the following. We know that for all $x \in \mathbb{R}$, $\tan^{-1} x$ is bounded. Clearly, $\tan^{-1} x \in (-\pi/2, \pi/2)$, $\forall x \in \mathbb{R}$. Hence, we have
$$\lim_{x \to \infty} f(x) = \infty, \qquad \lim_{x \to -\infty} f(x) = -\infty$$

This, along with the fact that f is a continuous function and hence, has the intermediate value property gives us that f is surjective. Hence, $f : \mathbb{R} \to \mathbb{R}$ is indeed, a bijection. [3]

(b) Let h be as defined above. From the expression for h, we see that for any values of c, h is continuous on the intervals $(-\infty, c)$ and (c, ∞). Hence, we only need to address the continuity of h at the point $x = c$. For h to be continuous at $x = c$, we need that $h(c) = \lim_{x \to c^-} h(x)$. Since f is monotonic, f^{-1} is monotonic. Thus f^{-1} is a monotonic mapping $\mathbb{R}$ onto $\mathbb{R}$. Now it follows that f^{-1} is a continuous function. Hence, we see that

$$\lim_{x \to c^-} h(x) = \lim_{x \to c^-} f^{-1}(x) = f^{-1}(c)$$

Hence, for h to be continuous, we need to find a value of $c \in \mathbb{R}$ such that $f(c) = f^{-1}(c)$. Hence, we must find c such that $f(f(c)) = c$. Calculating the left side from the expression for f gives us

$$\begin{aligned} & f(f(c)) = c + \tan^{-1} c + \tan^{-1}\left(c + \tan^{-1} c\right) = c \\ \implies \quad & \tan^{-1}\left(c + \tan^{-1} c\right) = \tan^{-1}(-c) \\ \implies \quad & c + \tan^{-1} c = -c \\ \implies \quad & 2c + \tan^{-1} c = 0 \end{aligned}$$

Note that in the above set of equations, we have used the fact that $\tan^{-1}$ is an odd, injective function. Looking at the final equation, we see that $c = 0$ is a solution. We also see that the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 2x + \tan^{-1} x$ is an increasing function. Clearly, g is differentiable and

$$g'(x) = 2 + \frac{1}{1 + x^2} > 0, \quad \forall x \in \mathbb{R}$$

Hence, we have that g is increasing. Hence, the equation $2c + \tan^{-1} c = g(c) = 0$ can have at most one solution. Since $c = 0$ is a solution, we can conclude that it is the only solution for the equation. Hence, the only value of c for which h is continuous is $c = 0$. [7]

(c) Since we want h to be differentiable, we must have h to be continuous. From the earlier part, this implies that $c = 0$. Hence, we only need to check the differentiability of h at $x = 0$. For h to be differentiable, we must have the left derivative of h to be equal to the right derivative of h. We have that

$$h'_+(0) = \lim_{t \to 0^+} \frac{h(t)}{t} = \lim_{t \to 0} \frac{f(t)}{t} = \lim_{t \to 0} \frac{t + \tan^{-1} t}{t} = 2$$

We also know that the derivative of the inverse function is the reciprocal of the derivative of the original function. Hence, we have that

$$h'_-(0) = (f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{2}$$

Since $h'_+(0) \neq h'_-(0)$, we conclude that for no real number c, is the function h differentiable at c. [3]

4. Let A, B be $n \times n$ ($n \geq 2$) nonsingular matrices with real entries.

(a) If $A^{-1} + B^{-1} = (A + B)^{-1}$, then prove that $\det A = \det B$.

(b) Find examples of matrices A, B satisfying $A^{-1} + B^{-1} = (A + B)^{-1}$.

(c) Find examples of matrices A, B with complex entries such that $A^{-1} + B^{-1} = (A + B)^{-1}$, but $\det A \neq \det B$. [13]

Solution:

(a) $(A+B)(A^{-1}+B^{-1}) = I$ implies $I + AB^{-1} + BA^{-1} = 0$. If $X = AB^{-1}$, then $I + X + X^{-1} = 0$. Hence $(X-I)X(I+X+X^{-1}) = 0$. Therefore $(X-I)(X^2 + X + I) = 0$ i.e. $X^3 = I$. Then $(\det X)^3 = 1$ i.e. $\det X = 1$. Hence $\det A = \det B$. [4]

(b) $A = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ OR any other correct example [5]

(c) $A = I$, $B = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$, where ω is a cube root of unity OR any other correct example [4]

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 06/01/2019 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are
   A) $k \leq 0$ B) $k \geq -4$ C) $k \geq 0$ or $k \leq -4$ D) $-4 \leq k \leq 0$.

2. Let $M_2(\mathbb{Z}_2)$ denote the set of all 2×2 matrices with entries from $\mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the set of integers modulo 2. The function $f : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ given by $f(x) = x^2$ is
   A) injective but not surjective B) bijective
   C) surjective but not injective D) neither injective nor surjective.

3. Consider the sequence $4, 0, 4.1, 0, 4.11, 0, 4.111, 0, \cdots$. This sequence
   A) converges to $4\frac{1}{9}$ B) has no convergent subsequence
   C) is unbounded D) is not convergent and has supremum $4\frac{1}{9}$.

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x-2)|(x-2)(x-3)|$. The function f is
   A) differentiable at $x = 2$ but not at $x = 3$ B) differentiable at $x = 3$ but not at $x = 2$
   C) differentiable at $x = 2$ and $x = 3$ D) neither differentiable at $x = 2$ nor at $x = 3$.

5. The equation $z^2 + \bar{z}^2 = 2$ represents the
   A) parabola B) pair of lines C) hyperbola D) ellipse.

6. The differential equation of the family of parabolas having their vertices at the origin and their foci on the X-axis is
   A) $2xdy - ydx = 0$ B) $xdy + ydx = 0$ C) $2ydx - xdy = 0$ D) $dy - xdx = 0$.

7. The number of solutions of the equation $\sqrt{1 - \sin x} = \cos x$ in $[0, 5\pi]$ is equal to
   A) 3 B) 6 C) 8 D) 11.

8. Consider $\triangle ABC$. Take 3 points on AB, 4 on BC and 5 on CA such that none of the points are vertices of $\triangle ABC$. The number of triangles that can be constructed using these points is
   A) 60 B) 205 C) 145 D) 120.

9. The number of primes p such that $p, p+10, p+14$ are all prime numbers is
   A) 0 B) 1 C) 3 D) infinitely many.

10. A relation R is defined on the set of positive integers as xRy if $2x + y \leq 5$. The relation R is
    A) reflexive B) symmetric C) transitive D) None of these.

Part II

N.B. Each question in Part II carries 6 marks.

1. Find all polynomials $p(x)$ such that $(p(x))^2 = 1 + xp(x+1)$ for all real numbers x.

2. A transposition of a vector X of length n is created by switching exactly two distinct entries of a vector X. For example, $(1,3,2,4)$ is a transposition of the vector $(1,2,3,4)$ of length 4. Find a vector X if it is given that all the vectors below are transpositions of X: $S = (0,1,1,1,0,0,0,1), T = (1,0,1,1,1,0,0,0), U = (1,0,1,0,1,0,0,1)$, $V = (1,1,1,1,0,0,0,0), W = (1,0,0,1,0,0,1,1)$.

3. In the complex plane, let u, v be two distinct solutions of $z^{2019} - 1 = 0$. Find the probability that $|u+v| \geq 1$.

4. Let $f : [a,b] \to [a,b]$ be a continuous function which is differentiable on (a,b) and $f(a) = a, f(b) = b$. Prove that there exist two distinct points x_1 and x_2 in (a,b) such that $f'(x_1)f'(x_2) = 1$.

5. Prove that there do not exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^{2018}$ and $g(f(x)) = x^{2019}$.

Part III

1. Let $f(x) = a_0x^n + \cdots + a_n$ be a non-constant polynomial with real coefficients satisfying

$$f(x)f(2x^2) = f(2x^3 + x)$$

for all real numbers x.

   (a) Show that $a_n \neq 0$.

   (b) Show that f has no real root.

   (c) Find a polynomial f satisfying $f(x)f(2x^2) = f(2x^3 + x)$ for all real numbers x.

[13]

2. For a subset $X = \{x_1, x_2, \cdots, x_n\}$ of the set of positive integers, $X + X$ denotes the set $\{x_i + x_j \ : \ i \neq j\}$ and $|X|$ denotes the number of elements in X.

   (a) Find subsets A, B of positive integers such that $|A| = |B| = 4, A \neq B$ and $A + A = B + B$.

   (b) Do there exist subsets A, B of positive integers such that $|A| = |B| = 3, A \neq B$ and $A + A = B + B$?

   (c) Show that if $n = 2^k$, then there exist subsets A, B of positive integers such that $|A| = |B| = n, A \neq B$ and $A + A = B + B$. [13]

3. On the real line place an object at 1. After every flip of a fair coin, move the object to the right by 1 unit if the outcome is Head and to the left by 1 unit if the outcome is Tail. Let N be a fixed positive integer. Game ends when the object reaches either 0 or N. Let $P(N)$ denote the probability of the object reaching N.

   (a) Find $P(3)$.

   (b) Find the formula for $P(N)$ for any positive integer N. [12]

4. Let f be a real valued differentiable function on $(0, \infty)$ satisfying

   (a) $|f(x)| \leq 5$ and

   (b) $f(x)f'(x) \geq \sin x$ for all $x \in (0, \infty)$.

   Does $\lim_{x \to \infty} f(x)$ exist? [12]

MADHAVA MATHEMATICS COMPETITION, January 6, 2019
Solutions and scheme of marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are
A) $k \leq 0$ B) $k \geq -4$ C) $k \geq 0$ or $k \leq -4$ D) $-4 \leq k \leq 0$.
Answer: C
Equating $kx = (x-1)^2$, we get $x^2 - (k+2)x + 1 = 0$. The discriminant is $(k+2)^2 - 4 = k(k+4) \geq 0$ for $k \geq 0$ or $k \leq -4$.

2. Let $M_2(\mathbb{Z}_2)$ denote the set of all 2×2 matrices with entries from $\mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the set of integers modulo 2. The function $f : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ given by $f(x) = x^2$ is
A) injective but not surjective B) bijective
C) surjective but not injective D) neither injective nor surjective.
Answer: D
The product $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ shows that the map is not injective and hence not surjective, since the domain and codomain are finite sets having same number of elements.

3. Consider the sequence $4, 0, 4.1, 0, 4.11, 0, 4.111, 0, \cdots$. This sequence
A) converges to $4\frac{1}{9}$ B) has no convergent subsequence
C) is unbounded D) is not convergent and has supremum $4\frac{1}{9}$.
Answer: D
$a_{2n} = 0 \ \ \forall n \in \mathbb{N}$. This is a convergent subsequence.(Option B not true)
a_{2n+1} is an increasing subsequence of positive real numbers. Also a_{2n+1} is bounded above by 5. Hence a_{2n+1} is convergent and converges to its supremum,
supremum of $a_{2n+1} = 4 + \sum_{n=1}^{\infty} \frac{1}{10^n} = 4\frac{1}{9}$.
Note that $\lim_{n\to\infty} a_{2n+1} = 4\frac{1}{9} \neq \lim_{n\to\infty} a_{2n} = 0$. Thus sequence is bounded (Option C not true) but it is not convergent (Option A incorrect). Clearly Option D is correct.

4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x-2)|(x-2)(x-3)|$. The function f is
A) differentiable at $x = 2$ but not at $x = 3$ B) differentiable at $x = 3$ but not at $x = 2$
C) differentiable at $x = 2$ and $x = 3$ D) neither differentiable at $x = 2$ nor at $x = 3$.
Answer: A
Using definition we get
$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$ and $\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} \neq \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3}$.
Thus the given function is differentiable at 2 but not differentiable at 3

5. The equation $z^2 + \bar{z}^2 = 2$ represents the
A) parabola B) pair of lines C) hyperbola D) ellipse.
Answer: C
The given equation reduces to $x^2 - y^2 = 1$, which represents a hyperbola.

6. The differential equation of the family of parabolas having their vertices at the origin and their foci on the X-axis is
A) $2xdy - ydx = 0$ B) $xdy + ydx = 0$ C) $2ydx - xdy = 0$ D) $dy - xdx = 0$.
Answer: A

Let $y^2 = ax$. Differentiating both sides we get, $2y\frac{dy}{dx} = a$. Substituting value of a, we get $2xdy - ydx = 0$.

7. The number of solutions of the equation $\sqrt{1-\sin x} = \cos x$ in $[0, 5\pi]$ is equal to
A) 3 B) 6 C) 8 D) 11.
Answer: B
Squaring both sides we get $1 - \sin x = 1 - \sin^2 x$ and thus $\sin x = 0$ or $\sin x = 1$. In $[0, 5\pi]$, we then have $x = 0, \pi, 2\pi, 3\pi, 4\pi, \pi/2, 5\pi/2, 7\pi/2$. We need to reject the values $\pi, 3\pi, 5\pi$ as in these cases LHS = 1 and RHS = -1. Hence number of solutions is 6.

8. Consider $\triangle ABC$. Take 3 points on AB, 4 on BC and 5 on CA such that none of the points are vertices of $\triangle ABC$. The number of triangles that can be constructed using these points is
A) 60 B) 205 C) 145 D) 120.
Answer: B
The triangles can be formed in the following ways
1) By choosing one point each on AB, BC, CA. The number of such triangles is 60.
2a) By choosing one point on AB and two on either BC or CA.
This can be done in $3\left[\binom{4}{2} + \binom{5}{2}\right]$.
2b) By choosing one point on BC and two on either AB or CA.
This can be done in $4\left[\binom{3}{2} + \binom{5}{2}\right]$.
2c) By choosing one point on AC and two on either AB or BC.
This can be done in $5\left[\binom{3}{2} + \binom{4}{2}\right]$.
Hence the total number of triangles that can be constructed using these points is 205.

9. The number of primes p such that $p, p+10, p+14$ are all prime numbers is
A) 0 B) 1 C) 3 D) infinitely many.
Answer: B
If $p = 3$, we note that 3,13,17 are all prime. Thus $p = 3$ is a solution. Any other prime is either of the type $3k+1$ or $3k+2$. If $p = 3k+1$ for some integer $k \geq 0$, then $p + 14 = 3(k+5)$ is not a prime. If $p = 3k+2$ for some integer $k \geq 0$, then $p + 10 = 3(k+4)$ is not a prime. Thus $p = 3$ is the only solution.

10. A relation R is defined on the set of positive integers as xRy if $2x + y \leq 5$. The relation R is
A) reflexive B) symmetric C) transitive D) None of these.
Answer: D
$(2, 2) \notin R$, so relation R is not reflexive.
Since $(1, 3) \in R$ but $(3, 1) \notin R$, so relation R is not symmetric.
Since $(2, 1) \in R$ and $(1, 3) \in R$ but $(2, 3) \notin R$, so relation R is not transitive.

Part II

N.B. Each question in Part II carries 6 marks.

1. Find all polynomials $p(x)$ such that $(p(x))^2 = 1 + xp(x+1)$ for all real numbers x.
Solution: Suppose $p(x)$ is a polynomial of degree n and it satisfies $(p(x))^2 = 1 + xp(x+1)$ for all real numbers x. Now, the degree of $(p(x))^2$ is $2n$ and the degree of $1 + xp(x+1)$ is $n+1$. Comparing left and right hand side of the given equation, we get $2n = n+1$. Therefore $n = 1$. Hence $p(x)$ is a linear polynomial. [3]
Let $p(x) = ax + b$, where $a \neq 0$. Substituting in the given equation, we get $(ax+b)^2 = 1 + x(a(x+1) + b)$. Simplifying this we get a quadratic equation, $(a^2 - a)x^2 + (2ab - a - b)x + (b^2 - 1) = 0$. This is true for all real numbers x. Hence

$a^2 - a = 0, 2ab - a - b = 0, b^2 - 1 = 0$. This implies that $a = 1, b = 1$. Therefore $p(x) = x + 1$. [3]

2. A transposition of a vector X of length n is created by switching exactly two distinct entries of a vector X. For example, $(1, 3, 2, 4)$ is a transposition of the vector $(1, 2, 3, 4)$ of length 4. Find a vector X if it is given that all the vectors below are transpositions of X: $S = (0, 1, 1, 1, 0, 0, 0, 1), T = (1, 0, 1, 1, 1, 0, 0, 0), U = (1, 0, 1, 0, 1, 0, 0, 1)$, $V = (1, 1, 1, 1, 0, 0, 0, 0), W = (1, 0, 0, 1, 0, 0, 1, 1)$.
Solution: Let $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$. The first observation is that X should have exactly 4 ones and 4 zeroes as all of S, T, U, V, W are of the same type. Then, the dot product of X with itself will count the number of ones occuring in the vector X. Hence, $\sum_{i=1}^{8} x_i^2 = 4$. Also, if we let Y denote the vector consisting of all ones, then $X \cdot Y = \sum_{i=1}^{8} x_i$ counts the number of ones present in the vector X and hence, $X \cdot Y = 4 = \sum_{i=1}^{8} x_i$. Now note that any permutation of X exchanges one zero and one one. Hence, the dot product of X with any of its transpositions has one less one i.e., $X \cdot S = X \cdot T = X \cdot U = X \cdot V = X \cdot W = 3$. This gives rise to the following system of equations:

$$x_2 + x_3 + x_4 + x_8 = 3,$$
$$x_1 + x_3 + x_4 + x_5 = 3,$$
$$x_1 + x_3 + x_5 + x_8 = 3,$$
$$x_1 + x_2 + x_3 + x_4 = 3,$$
$$x_1 + x_4 + x_7 + x_8 = 3.$$

[3]

Solving the above system, it is easy to check that we get $x_1 = x_4 = x_8$ and $x_2 = x_5$. These equations together with $x_1 + x_4 + x_7 + x_8 = 3$ give that $x_7 = 0$ and $x_1 = x_4 = x_8 = 1$. Since $4 = \sum_{i=1}^{8} x_i$, we get $x_2 + x_3 + x_5 + x_6 + x_7 = 1$. Now if both $x_2 = x_5$ are one, substituting in the above equation will give a contradiction. Hence, $x_2 = x_5 = 0$. From this it is easy to deduce that $x_6 = x_7 = 0, x_3 = 1$. Thus, we get $X = (1, 0, 1, 1, 0, 0, 0, 1)$. [3]

3. In the complex plane, let u, v be two distinct solutions of $z^{2019} - 1 = 0$. Find the probability that $|u + v| \geq 1$.
Solution: Let u, v be two distinct solutions of $z^n - 1 = 0$. Then we can write $u = e^{i2\pi k/n}, v = e^{i2\pi m/n}, m \neq k$.
Observe that $|u + v| = |u||1 + \frac{v}{u}| = |1 + e^{i2\pi(m-k)/n}|$
$= |1 + \cos(2\pi(m-k)/n) + i\sin(2\pi(m-k)/n)| = \sqrt{2 + 2\cos(2\pi(m-k)/n)}$
Now $|u + v| \geq 1$ if and only if $\frac{-1}{2} \leq \cos\frac{2\pi(m-k)}{n}$. [3]
This is true if and only if $\frac{-2\pi}{3} \leq \frac{2\pi(m-k)}{n} \leq \frac{2\pi}{3}$.
This is true if and only if $\frac{-1}{3} \leq \frac{m-k}{n} \leq \frac{1}{3}$.

For $n = 2019$, there are 2×673 possibilities of m for each k. Hence the required probability is $\frac{2 \times 673 \times 2019}{2019 \times 2018} = \frac{2 \times 673}{2018}$. [3]

4. Let $f : [a, b] \to [a, b]$ be a continuous function which is differentiable on (a, b) and $f(a) = a, f(b) = b$. Prove that there exist two distinct points x_1 and x_2 in (a, b) such that $f'(x_1)f'(x_2) = 1$.
Solution: Let $g = f \circ f$. Then $g(a) = a, g(b) = b$. By Lagrange's Mean Value Theorem,

there exists $c \in (a,b)$ such that $\dfrac{g(b)-g(a)}{b-a} = 1 = g'(c)$.
This implies that $f'(f(c))f'(c) = 1$. [3]
Case 1: If $f(c) \neq c$, then choose $x_1 = c$ and $x_2 = f(c)$. [1]
Case 2: Let $f(c) = c$. Applying Lagrange's Mean Value Theorem to f on intervals $[a,c]$ and $[c,b]$, we get $x_1 \in (a,c)$ and $x_2 \in (c,b)$ such that

$$\frac{f(c)-f(a)}{c-a} = 1 = f'(x_1), \quad \frac{f(b)-f(c)}{b-c} = 1 = f'(x_2).$$

Hence $f'(x_1)f'(x_2) = 1$. [2]

5. Prove that there do not exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^{2018}$ and $g(f(x)) = x^{2019}$.
Solution: Suppose there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^{2018}$ and $g(f(x)) = x^{2019}$. Therefore $(f \circ g)(f(x)) = (f(x))^{2018}$. Since the composition is associative, we have $f \circ (g \circ f)(x) = (f(x))^{2018}$. This implies that

$$f(x^{2019}) = (f(x))^{2018}.$$

[3]
Since 2019 is an odd number, $g \circ f$ is an injective function. Therefore f is also injective. Substituting $x = 0, 1, -1$ in the above equation, we get

$$(f(0))^{2018} = f(0), (f(1))^{2018} = f(1), (f(-1))^{2018} = f(-1).$$

But, only real solutions of $x^{2018} = x$ are 0 and 1. This implies that at least two of $f(0), f(1), f(-1)$ are same which contradicts the injectivity of the function f. Hence there do not exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^{2018}$ and $g(f(x)) = x^{2019}$.
[3]

Part III

1. Let $f(x) = a_0x^n + \cdots + a_n$ be a non-constant polynomial with real coefficients satisfying

$$f(x)f(2x^2) = f(2x^3 + x)$$

for all real numbers x.

(a) Show that $a_n \neq 0$.
(b) Show that f has no real root.
(c) Find a polynomial f satisfying $f(x)f(2x^2) = f(2x^3 + x)$ for all real numbers x.
[13]

Solution:

(a) Let k be the greatest index such that $a_k \neq 0$. Then the left hand side has a form $f(x)f(2x^2) = a_0^2 2^n x^{3n} + \cdots + a_k^2 2^{n-k} x^{3(n-k)}$ and the right hand side has a form $f(2x^3 + x) = a_0 2^n x^{3n} + \cdots + a_k x^{n-k}$. So we must have

$$a_k^2 2^{n-k} x^{3(n-k)} = a_k x^{n-k}, \forall x \in \mathbb{R},$$

which gives $n = k$, that is $a_n = a_k \neq 0$. [4]

(b) Suppose $x_0 \neq 0$ is a root of $f(x)$. Consider a sequence

$$x_{n+1} = 2x_n^3 + x_n, n \geq 0.$$

Note that if $x_0 > 0$, then $\{x_n\}$ is increasing and if $x_0 < 0$, then $\{x_n\}$ is decreasing. From the assumption of the problem, it follows that if $f(x_0) = 0$ with $x_0 \neq 0$, then $f(x_k) = 0, \forall k$. This shows that a non-constant polynomial of degree n has infinitely many roots, which is impossible. Thus f has no real root. [6]

(c) $f(x) = x^2 + 1$ [3]

2. For a subset $X = \{x_1, x_2, \cdots, x_n\}$ of the set of positive integers, $X + X$ denotes the set $\{x_i + x_j \ : \ i \neq j\}$ and $|X|$ denotes the number of elements in X.

   (a) Find subsets A, B of positive integers such that $|A| = |B| = 4, A \neq B$ and $A + A = B + B$.

   (b) Do there exist subsets A, B of positive integers such that $|A| = |B| = 3, A \neq B$ and $A + A = B + B$?

   (c) Show that if $n = 2^k$, then there exist subsets A, B of positive integers such that $|A| = |B| = n, A \neq B$ and $A + A = B + B$. [13]

   Solution:

   (a) $A = \{1, 4, 6, 7\}, B = \{2, 3, 5, 8\}$. [3]

   (b) Let $A = \{a, b, c\}$ and $B = \{d, e, f\}$. Suppose $a < b < c$ and $d < e < f$. Suppose $A + A = B + B$. Now by the given condition, we have

   $$a + b = d + e, b + c = e + f, a + c = d + f.$$

   Adding all these we get, $a + b + c = d + e + f$. This implies $a = d, b = e, c = f$. This is contradiction. [4]

   (c) Note that if sets $A = \{a_1, \cdots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ have property that $|A| = |B| = n, A \neq B$ and $A+A = B+B$, then for a suitable positive integert m, the sets $A' = \{a_1, \ldots, a_n, b_1+m, \cdots, b_n+m\}$ and $B' = \{a_1+m, \ldots, a_n+m, b_1, \ldots, b_n\}$ will have property that $|A'| = |B'| = 2n, A' \neq B'$ and $A' + A' = B' + B'$. Now starting with $A = \{1, 4\}$ and $B = \{2, 3\}$, for any positive integert k, we can inductively find the sets of sizes 2^k with desired property. [6]

3. On the real line place an object at 1. After every flip of a fair coin, move the object to the right by 1 unit if the outcome is Head and to the left by 1 unit if the outcome is Tail. Let N be a fixed positive integer. Game ends when the object reaches either 0 or N. Let $P(N)$ denote the probability of the object reaching N.

   (a) Find $P(3)$.

   (b) Find the formula for $P(N)$ for any positive integer N. [12]

   Solution:

   (a) Starting the game at 1, the possible outcomes to reach 3 without reaching zero are HH, HTHH, HTHTHH and so on. Hence the probability of reaching 3 without going to zero is given by a geometric series $\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$ which adds up to $\frac{1}{3}$. Hence $P(3) = \frac{1}{3}$. [6]

   (b) For any positive integer N, $P(N) = P(N-1)[P(N-1 \to N)]$ where $[P(N-1 \to N)]$ denotes probabilty of reaching N from $N-1$ without reachin zero. Now by symmetry, $[P(N-1 \to N)]$ is equal to $[P(1 \to 0)]$ without reaching N. But $[P(1 \to 0)]$ is equal to $1 - P(N)$. Thus we have a recurrence relation $P(N) = P(N-1)(1 - P(N))$. Now with $P(1) = 1, P(2) = \frac{1}{2}, P(3) = \frac{1}{3}$, the recurrence relation obtained above allows us to prove that $P(N) = \frac{1}{N}$. [6]

4. Let f be a real valued differentiable function on $(0, \infty)$ satisfying

   (a) $|f(x)| \leq 5$ and

(b) $f(x)f'(x) \geq \sin x$ for all $x \in (0, \infty)$.

Does $\lim_{x\to\infty} f(x)$ exist? [12]

Solution: Consider $F(x) = f^2(x) + 2\cos x$ defined on $(0, \infty)$. Then by (a),

$$|F(x)| \leq |f(x)|^2 + 2|\cos x| \leq 5^2 + 2.$$

By (b), $F'(x) = 2f(x)f'(x) - 2\sin x \geq 0$, $\forall x \in (0, \infty)$ implying that $F(x)$ is an increasing function. [6]

Let $\{x_n\} := \{2\pi, 2\pi + \frac{\pi}{2}, 4\pi, 4\pi + \frac{\pi}{2}, 6\pi, 6\pi + \frac{\pi}{2}, \cdots\}$. Observe that $\forall n,\ x_n > 0, \{x_n\}$ is an increasing sequence and $x_n \to \infty$ as $n \to \infty$.

Put $u_n = F(x_n)$. Now observe that $\{u_n\}$ is an increasing sequence which is bounded above. Thus $\{u_n\}$ is convergent. Assume that $\lim_{x\to\infty} f(x)$ exists. This implies that if $v_n = f(x_n)$, then $\lim_{n\to\infty} v_n$ exists. Therefore $\lim_{n\to\infty}[F(x_n) - f^2(x_n)]$ exists. Now

$$\lim_{n\to\infty}[F(x_n) - f^2(x_n)] = \lim_{n\to\infty} u_n - \lim_{x\to\infty} v_n^2.$$

This implies that $\lim_{n\to\infty} \cos x_n$ exists. This is not possible because the sequence $\{\cos x_n\} = \{1, 0, 1, 0, \cdots\}$ has no limit. [6]

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 12/01/2020 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let A be a non-empty subset of real numbers and $f : A \to A$ be a function such that $f(f(x)) = x$ for all $x \in A$. Then $f(x)$ is
   A) a bijection B) one-one but not onto
   C) onto but not one-one D) neither one-one nor onto.

2. If $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x+y) = f(xy)$ for all $x, y \in \mathbb{R}$ and $f(3/4) = 3/4$, then $f(9/16) =$
   A) 9/16 B) 0 C) 3/2 D) 3/4.

3. The area enclosed between the curves $y = \sin^2 x$ and $y = \cos^2 x$ in the interval $0 \le x \le \pi/2$ is
   A) 2 B) 1/2 C) 1 D) 3/4.

4. The number of ordered pairs (m, n) of all integers satisfying $\dfrac{m}{12} = \dfrac{12}{n}$ is
   A) 15 B) 30 C) 12 D) 10.

5. Suppose $2\log x + \log y = x - y$. Then the equation of the tangent line to the graph of this equation at the point $(1, 1)$ is
   A) $x + 2y = 3$ B) $x - 2y = 3$ C) $2x + y = 3$ D) $2x - y = 3$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sin[x]$, where $[x]$ denotes the greatest integer less than or equal to x. Then
   A) f is a 2π-periodic function B) f is a π-periodic function
   C) f is a 1-periodic function D) f is not a periodic function.

7. For how many integers a with $1 \le a \le 100$, a^a is a square?
   A) 50 B) 51 C) 55 D) 56.

8. $\lim\limits_{x \to 0} x \left[\dfrac{1}{x}\right]$
   A) 0 B) 1 C) -1 D) does not exist.

9. If α and β are the roots of $x^2 + 3x + 1$ then $\left(\dfrac{\alpha}{\beta+1}\right)^2 + \left(\dfrac{\beta}{\alpha+1}\right)^2$ equals
   A) 19 B) 18 C) 20 D) 17.

10. The equation $z^3 + iz - 1 = 0$ has
    A) no real root B) exactly one real root
    C) three real roots D) exactly two real roots.

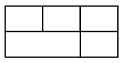
Part II

N.B. Each question in Part II carries 6 marks.

1. Let $a_1, a_2, \cdots$ be a sequence of natural numbers. Let (a, b) denote the greatest common divisor (gcd) of a and b. If $(a_m, a_n) = (m, n)$ for all $m \neq n$, prove that $a_n = n$ for all $n \in \mathbb{N}$.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(z)f(iz) = z^2$ for all $z \in \mathbb{C}$. Prove that $f(z) + f(-z) = 0$ for all $z \in \mathbb{C}$. Find such a function.

3. Let n be a positive integer. Line segments can be drawn parallel to edges of a given rectangle. What is the minimum number of line segments (not necessarily of same lengths) that are required so as to divide the rectangle into n subrectangles? Justify.


For example, in the adjacent figure, 3 segments are drawn to get 5 subrectangles and 3 is the minimum number.

4. Let $f : [0, 1] \to (0, \infty)$ be a continuous function satisfying $\displaystyle\int_0^1 f(t)dt = \frac{1}{3}$. Show that there exists $c \in (0, 1)$ such that $\displaystyle\int_0^c f(t)dt = c - \frac{1}{2}$.

5. Let $A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$. Show that there exist matrices X, Y such that $A = X^3 + Y^3$.

Part III

1. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function satisfying $f(1) = 5$ and $f\left(\frac{x}{x+1}\right) = f(x) + 2$ for all positive real numbers x.
   a) Find $\lim_{x\to\infty} f(x)$.
   b) Show that $\lim_{x\to 0^+} f(x) = \infty$.
   c) Find one example of such a function. [12]

2. An $n \times n$ matrix $A = (a_{ij})$ is given. The sum of any n entries of A, whose any two entries lie on different rows and different columns, is the same.
   a) Prove that there exist numbers $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ such that $a_{ij} = x_i + y_j$ for all $i, j,\ \ 1 \leq i, j \leq n$.
   b) Prove that $\text{rank}(A) \leq 2$. [12]

3. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function. Let
$$J = \left\{ \frac{f(b) - f(a)}{b - a} \ : \ a, b \in I, a < b \right\}.$$
   Show that a) J is an interval.
   b) $J \subseteq f'(I)$ and $f'(I) - J$ contains at most two elements. [13]

4. Let q, n be positive integers such that $1 < q < n$ and $\gcd(q, n) = 1$.
   a) Show that there exist unique integers k, r such that $n = kq - r,\ 0 \leq r < q$.
   b) Show that there exists a unique positive integer m and unique integers $b_1, b_2, \cdots, b_m$ all ≥ 2 satisfying $\dfrac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \ddots - \cfrac{1}{b_{m-1} - \cfrac{1}{b_m}}}}$.
   c) If $b_j > 2$ for some j, then show that $\displaystyle\sum_{i=1}^{m} (b_i - 2) < 2(n - q - 1)$. [13]

MADHAVA MATHEMATICS COMPETITION, 12th January 2020
Solutions and scheme of marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let A be a non-empty subset of real numbers and $f : A \to A$ be a function such that $f(f(x)) = x$ for all $x \in A$. Then $f(x)$ is
   A) a bijection B) one-one but not onto
   C) onto but not one-one D) neither one-one nor onto.
   Answer: A
   If $f(x) = f(y)$, then $f(f(x)) = f(f(y))$ implies $x = y$. Therefore f is one-one function. By definition, f is onto. Hence f is a bijection.

2. If $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x + y) = f(xy)$ for all $x, y \in \mathbb{R}$ and $f(3/4) = 3/4$, then $f(9/16) =$
   A) 9/16 B) 0 C) 3/2 D) 3/4.
   Answer: D
   If we put $y = 0$, then $f(x) = f(0)$ for all $x \in \mathbb{R}$. This implies that f is a constant function.

3. The area enclosed between the curves $y = \sin^2 x$ and $y = \cos^2 x$ in the interval $0 \le x \le \pi/2$ is
   A) 2 B) 1/2 C) 1 D) 3/4.
   Answer: C
   Area $= \displaystyle\int_0^{\pi/4} (\cos^2 x - \sin^2 x)dx + \int_{\pi/4}^{\pi/2} (\sin^2 x - \cos^2 x)dx$
   $= \displaystyle\int_0^{\pi/4} \cos 2x dx - \int_{\pi/4}^{\pi/2} \cos 2x dx = 1/2 + 1/2 = 1.$

4. The number of ordered pairs (m, n) of all integers satisfying $\dfrac{m}{12} = \dfrac{12}{n}$ is
   A) 15 B) 30 C) 12 D) 10.
   Answer: B
   We have $mn = 144$ and $144 = 2^4 3^2$. There are 15 divisors of 144 which are positive integers. Including negative integers total 30 pairs are there.

5. Suppose $2\log x + \log y = x - y$. Then the equation of the tangent line to the graph of this equation at the point $(1, 1)$ is
   A) $x + 2y = 3$ B) $x - 2y = 3$ C) $2x + y = 3$ D) $2x - y = 3$.
   Answer: A
   Consider $2\log x + \log y = x - y$. Differentiating this equation with respect to x,

we get $\frac{2}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$. At point $(1,1)$ the slope of tangent is $-1/2$. The equation of tangent is $x + 2y = 3$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sin[x]$, where $[x]$ denotes the greatest integer less than or equal to x. Then
   A) f is a 2π-periodic function B) f is a π-periodic function
   C) f is a 1-periodic function D) f is not a periodic function.
   Answer: D
   If f is a periodic function with period T, then $\sin[x+T] = \sin[x]$. This implies $[x+T] - [x] = 2n\pi$ for some integer n, which is not possible.

7. For how many integers a with $1 \le a \le 100$, a^a is a square?
   A) 50 B) 51 C) 55 D) 56.
   Answer: C
   Case 1: all even integers. There are 50 even integers between 1 and 100.
   Case 2: a is an odd number which is a square i. e. 1,9,25,49,81.
   Therefore total number is 55.

8. $\lim_{x\to 0} x\left[\frac{1}{x}\right]$
   A) 0 B) 1 C) -1 D) does not exist.
   Answer: B
   $\lim_{x\to 0}\left(x\left[\frac{1}{x}\right] - 1\right) = \lim_{x\to 0}\left(x\left[\frac{1}{x}\right] - x\frac{1}{x}\right) = \lim_{x\to 0} x\left(\left[\frac{1}{x}\right] - \frac{1}{x}\right) = 0.$

9. If α and β are the roots of $x^2 + 3x + 1$ then $\left(\frac{\alpha}{\beta+1}\right)^2 + \left(\frac{\beta}{\alpha+1}\right)^2$ equals
   A) 19 B) 18 C) 20 D) 17.
   Answer: B
   Observe that $\alpha + \beta = -3, \alpha\beta = 1, \alpha^2 + 3\alpha + 1 = 0, \beta^2 + 3\beta + 1 = 0$. Therefore $(\alpha+1)^2 = \alpha^2 + 2\alpha + 1 = -\alpha, (\beta+1)^2 = \beta^2 + 2\beta + 1 = -\beta$.
   $\left(\frac{\alpha}{\beta+1}\right)^2 + \left(\frac{\beta}{\alpha+1}\right)^2 = \left(\frac{\alpha^2}{-\beta}\right) + \left(\frac{\beta^2}{-\alpha}\right) = \frac{1+3\alpha}{\beta} + \frac{1+3\beta}{\alpha}$
   $= \frac{(1+3\alpha)\alpha + (1+3\beta)\beta}{\alpha\beta} = 3(\alpha^2+\beta^2) + (\alpha+\beta) = 18.$

10. The equation $z^3 + iz - 1 = 0$ has
    A) no real root B) exactly one real root
    C) three real roots D) exactly two real roots.
    Answer: A
    If x is a real root, then $x^3 + ix - 1 = 0$. This implies that $x = 0$ and $x^3 = 1$, which is not possible.

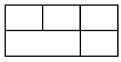
Part II

N.B. Each question in Part II carries 6 marks.

1. Let $a_1, a_2, \cdots$ be a sequence of natural numbers. Let (a, b) denote the greatest common divisor (gcd) of a and b. If $(a_m, a_n) = (m, n)$ for all $m \neq n$, prove that $a_n = n$ for all $n \in \mathbb{N}$.
   Solution: Note that $(a_n, a_{n^2}) = (n, n^2) = n$. Therefore n divides a_n. [2]
   Let $a_n = mn$. Then $(a_n, a_{mn}) = (n, mn) = n$ and $(a_m, a_{mn}) = (m, mn) = m$. [2]
   This implies that mn divides a_{mn}. Since mn divides both a_{mn} and a_n, it divides their gcd n. Hence $m = 1$ and thus $a_n = n$. [2]

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(z)f(iz) = z^2$ for all $z \in \mathbb{C}$. Prove that $f(z) + f(-z) = 0$ for all $z \in \mathbb{C}$. Find such a function.
   Solution: It is given that $f(z)f(iz) = z^2$ for all $z \in \mathbb{C}$. Replacing z by iz, we get $f(iz)f(-z) = -z^2$. Adding these two expressions we get, $f(iz)[f(z)+f(-z)] = 0$.
   From $f(z)f(iz) = z^2$ we deduce that $f(z) = 0$ if and only if $z = 0$.
   If $z \neq 0$, then $f(iz) \neq 0$ and so $f(z) + f(-z) = 0$.
   If $z = 0$, then $f(z) + f(-z) = 2f(0) = 0$. Thus $f(z) + f(-z) = 0,\ \forall z \in \mathbb{C}$. [4]
   Example: $f(z) = \left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) z$ or $f(z) = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) z$ [2]

3. Let n be a positive integer. Line segments can be drawn parallel to edges of a given rectangle. What is the minimum number of line segments (not necessarily of same lengths) that are required so as to divide the rectangle into n subrectangles? Justify.

   
   For example, in the adjacent figure, 3 segments are drawn to get 5 subrectangles and 3 is the minimum number.
   Solution: It can be observed that there are three cases.
   Case 1: Let $n = k^2$. Consider $(k-1)$ horizontal and $(k-1)$ vertical line segments. These $2(k-1)$ line segments will generate k^2 subrectangles. Note that this is a configuration with minimum number of line segments to divide the rectangle into k^2 subrectangles. [2]
   Case 2: Let $k^2 + 1 \leq n \leq k^2 + k$. Consider a small horizontal line segment, which divides one of k^2 subrectangles into 2 subrectangles resulting into total $k^2 + 1$ subrectangles. Extending this small line segment, we can get up to $k^2 + k$ subrectangles. Hence, in this case the the minimum number of line segments required to divide the rectangle into n subrectangles is $2(k-1) + 1 = 2k - 1$. [3]
   Case 3: Let $k^2 + k + 1 \leq n \leq k^2 + 2k + 1$. Applying the same procedure as given in case 2, but instead of horizontal line segment we need to take a vertical line segment and extend it as above. Thus, in this case the minimum number of line segments required to divide the rectangle into n subrectangles is $(2k-1)+1 = 2k$. [1]

4. Let $f : [0,1] \to (0,\infty)$ be a continuous function satisfying $\int_0^1 f(t)dt = \frac{1}{3}$. Show that there exists $c \in (0,1)$ such that $\int_0^c f(t)dt = c - \frac{1}{2}$.

Solution: Define $g(x) = \int_0^x f(t)dt + \frac{1}{2}$. [2]

Then $g : [0,1] \to [0,1]$ is a continuous function. By fixed point theorem, there exists $c \in (0,1)$ such that $g(c) = c$. Observe that $c \neq 0, 1$. Thus we get $\int_0^c f(t)dt = c - \frac{1}{2}$. [4]

5. Let $A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$. Show that there exist matrices X, Y such that $A = X^3 + Y^3$.

Solution: Note that $A^2 + 3A + 2I = 0$. [2]

Therefore $A(A^2 + 3A + 2I) = A^3 + 3A^2 + 2A = 0$. This implies that $(A+I)^3 = A^3 + 3A^2 + 3A + I = A + I$. [2]

Hence $A = (A+I)^3 - I = (A+I)^3 + (-I)^3$.

Thus we get $X = A + I = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ and $Y = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. [2]

Part III

1. Let $f : (0,\infty) \to \mathbb{R}$ be a continuous function satisfying $f(1) = 5$ and $f\left(\frac{x}{x+1}\right) = f(x) + 2$ for all positive real numbers x.

a) Find $\lim_{x\to\infty} f(x)$.

b) Show that $\lim_{x\to 0^+} f(x) = \infty$.

c) Find one example of such a function. [12]

Solution: a) Note that as $x \to \infty$, $\frac{x}{x+1} \to 1$.

Hence $\lim_{x\to\infty} f(x) = f(1) - 2 = 5 - 2 = 3$. [2]

b) Observe that as $x \to 0^+$, $\frac{x}{x+1} \to 0$. If $\lim_{x\to 0^+} f(x) = L$, then $L = L + 2$. This implies that the limit is not finite. [1]

Define $g : [\frac{1}{n+1}, \frac{1}{n}] \to [\frac{1}{n+2}, \frac{1}{n+1}]$ as $g(x) = \frac{x}{x+1}$. Note that $g(\frac{1}{n+1}) = \frac{1}{n+2}$, $g(\frac{1}{n}) = \frac{1}{n+1}$ and $g'(x) = \frac{1}{(x+1)^2} > 0$. Therefore g is an increasing function and hence g is one-one and onto. For $x \in [\frac{1}{3}, \frac{1}{2}]$, there is some $t \in [\frac{1}{2}, 1]$ such that $f(x) = f(g(t)) = f(\frac{t}{t+1}) = f(t) + 2$. By induction, it can be proved that for $x \in [\frac{1}{n+1}, \frac{1}{n}]$, there is some $t \in [\frac{1}{2}, 1]$ such that $f(x) = f(t) + 2(n-1)$.

Let $M > 0$ be any real number. Since f is continuous on $[\frac{1}{2}, 1]$, it is bounded and attains its bounds. There exists $M_1 > 0$ such that $-M_1 < f(t) < M_1$, $\forall t \in [\frac{1}{2}, 1]$. Suppose $f(t_0) = \min f(t)$, $t \in [\frac{1}{2}, 1]$. Therefore $f(t) \geq f(t_0)$, $\forall t \in [\frac{1}{2}, 1]$. Choose $n_1 \in \mathbb{N}$ such that $f(t_0) + 2(n_1 - 1) > 0$. We can choose $n_2 \in \mathbb{N}$ such that

$n_2 > M + M_1 + 2$. Suppose $n_0 = \max\{n_1, n_2\}$ and $\delta = \frac{1}{n_0} > 0$. If $0 < x < \delta$, then $x \in [\frac{1}{n_0+1}, \frac{1}{n_0}]$ and $|f(x)| = |f(t) + 2(n_0 - 1)|,\ t \in [\frac{1}{2}, 1]$.
Now $f(t) + 2(n_0 - 1) \geq f(t_0) + 2(n_0 - 1) \geq f(t_0) + 2(n_1 - 1) > 0$.
Note that $|f(x)| = f(t) + 2(n_0 - 1) \geq -M_1 + 2(n_0 - 1) > -M_1 - 2 + n_0 \geq -M_1 - 2 + n_2 > M$.
Hence $\lim_{x \to 0^+} f(x) = \infty$. [6]

c) Example: $f(x) = 3 + \dfrac{2}{x}$. [3]

2. An $n \times n$ matrix $A = (a_{ij})$ is given. The sum of any n entries of A, whose any two entries lie on different rows and different columns, is the same.
a) Prove that there exist numbers $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ such that $a_{ij} = x_i + y_j$ for all $i, j,\ \ 1 \leq i, j \leq n$.
b) Prove that rank$(A) \leq 2$. [12]

Solution: a) Consider n entries situated on different rows and different columns $a_{ij_i}, i = 1, 2, \cdots, n$. Fix k and $l, 1 \leq k < l \leq n$ and replace a_{kj_k} and a_{lj_l} with a_{kj_l} and a_{lj_k} respectively. The new n entries are still situated on different rows and different columns. Since the sums of sets of n entries are equal, this implies
$a_{kj_k} + a_{lj_l} = a_{kj_l} + a_{lj_k}$. [*]
Now denote by $x_1, x_2, \cdots, x_n$ the entries in the first column and by $x_1, x_1+y_2, x_1+y_3, \cdots, x_1 + y_n$ the entries in the first row.

$$A = \begin{pmatrix} x_1 & x_1 + y_2 & x_1 + y_3 & \cdots & x_1 + y_n \\ x_2 & \cdots & & & \\ \vdots & & & & \\ x_n & \cdots & & & \end{pmatrix}$$

That is we have defined $x_k = a_{k1}$ for all $k, y_1 = 0$ and $y_k = a_{1k} - a_{k1}$ for all $k \geq 2$. Now $a_{ij} = x_i + y_j$ for all i, j with $i = 1$ or $j = 1$. Consider $i, j > 1$. From [*] we deduce that $a_{11} + a_{ij} = a_{1j} + a_{i1}$. Hence $x_1 + a_{ij} = x_i + x_1 + y_j$. This implies $a_{ij} = x_i + y_j$ for all i, j. [8]
b) Let A_j denote the j^{th} column of matrix A. Then

$$A_j = \begin{pmatrix} x_1 + y_j \\ x_2 + y_j \\ \vdots \\ x_n + y_j \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + y_j \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \alpha + y_j\beta, \text{ where } \alpha = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

That is $A_j \in \langle \alpha, \beta \rangle$, where $\langle \alpha, \beta \rangle$ denotes the linear span of α, β. This is true for every column of A. Hence rank$(A) \leq \dim \langle \alpha, \beta \rangle \leq 2$. [4]

3. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a differentiable function. Let

$$J = \left\{ \frac{f(b) - f(a)}{b - a} \ : \ a, b \in I, a < b \right\}.$$

Show that a) J is an interval.
b) $J \subseteq f'(I)$ and $f'(I) - J$ contains at most two elements. [13]

Solution: a) Note that J is an interval if and only if for every $a, b \in J, a < b$, we have $(a, b) \subset J$. Let $u, v \in J, u < v$. Hence $u = \dfrac{f(b_1) - f(a_1)}{b_1 - a_1}, v = \dfrac{f(b_2) - f(a_2)}{b_2 - a_2}$. Suppose $p \in (u, v)$. Define a function Q on $[0, 1]$ as

$$Q(t) = \frac{f(tb_1 + (1-t)b_2) - f(ta_1 + (1-t)a_2)}{t(b_1 - a_1) + (1-t)(b_2 - a_2)}.$$

Since Q is a continuous function and $Q(0) = v, Q(1) = u$, we deduce that there exists t_0 such that $Q(t_0) = p$. Hence $p = \dfrac{f(b_0) - f(a_0)}{b_0 - a_0}$, where $b_0 = t_0 b_1 + (1-t_0)b_2$ and $a_0 = t_0 a_1 + (1 - t_0)a_2$. Thus $p \in J$. [6]

b) Using Lagrange's Mean Value Theorem, we get $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ for some $c \in (a, b)$. Hence $J \subseteq f'(I)$. [2]

Let $p \in I$ and $y_n = \dfrac{f(p + (1/n)) - f(p)}{1/n}$. If p is the right end point of an interval I, then choose $y_n = \dfrac{f(p) - f(p - (1/n))}{1/n}$. Note that y_n converges to $f'(p)$. Hence $f'(p) \in \overline{J}$. Therefore $f'(I) \subseteq \overline{J}$. Since $J \subseteq f'(I) \subseteq \overline{J}$ and J being an interval, the set $f'(I) - J$ contains at most two elements, which are the end points of J. [5]

4. Let q, n be positive integers such that $1 < q < n$ and $\gcd(q, n) = 1$.
a) Show that there exist unique integers k, r such that $n = kq - r,\ 0 \leq r < q$.
b) Show that there exists a unique positive integer m and unique integers $b_1, b_2, \cdots, b_m$ all ≥ 2 satisfying $\dfrac{n}{q} = b_1 - \dfrac{1}{b_2 - \frac{1}{b_3 - \ddots - \frac{1}{b_{m-1} - \frac{1}{b_m}}}}$.
c) If $b_j > 2$ for some j, then show that $\displaystyle\sum_{i=1}^{m} (b_i - 2) < 2(n - q - 1)$. [13]

Solution:

(a) By division algorithm, there exists unique $k_1, r_1 \in \mathbb{N}$ such that $n = k_1 q + r_1$, where $0 \leq r_1 < q$. Further, $\gcd(q, n) = 1 \implies r_1 > 0$.
Thus, we get, $n = (k_1 + 1)q - (q - r_1)$. Since, $r_1 < q$, we have, $r = q - r_1 > 0,\ r < q$ and let $k = k_1 + 1 \in \mathbb{N}$.
Hence, there exists unique $k, r \in \mathbb{N}$ such that $n = kq - r$, where $0 < r < q$. [3]

(b) By (a), we get $k, r \in \mathbb{N}$ such that $n = kq - r$ and $0 < r < q$.
Thus, $\dfrac{n}{q} = k - \dfrac{r}{q}$. Let $b_1 = k$. Now, $n > q \implies k \geq 2 \implies b_1 \geq 2$.

Observe : $\frac{n}{q} = b_1 - \frac{1}{q/r}$. Now, $q > r$; $\gcd(q, r) = 1$. Let $r_1 = r$. Hence, by applying the above procedure to q and r_1, we get,

$$\frac{q}{r_1} = b_2 - \frac{1}{r_1/r_2}$$

where, $b_2 \geq 2$ and $r_2 < r_1$.
Repeating these steps, we get, $r_1, r_2, \ldots, r_m$ such that $0 < r_m < r_{m-1} < r_{m-2} < \cdots < r_2 < r_1$. The process ends after finitely many steps(say m steps), with $r_m = 1$.
Now, we prove that, whenever, in the above expression, $b_i \geq 2$, for each $i = 1, 2, \ldots, m$ then such b_is are unique. ..... (I)
In fact, we prove that if $b_i \geq 2$, for all $i = 1, 2, \ldots, m$ then $b_1 = \left\lfloor \frac{n}{q} \right\rfloor + 1$
Case (i) if $b_1 > \left\lfloor \frac{n}{q} \right\rfloor + 1$.
In this case, we can write: $\frac{n}{q} = b_1 - \frac{r}{q}$, for some $r > q$.
Thus, $\frac{n}{q} = b_1 - \frac{1}{q/r} = b_1 - \frac{1}{b_2 - s}$; where $b_2 - s = \frac{q}{r}$, giving $s = b_2 - \frac{q}{r}$
Now, we have, $r > q$ and $b_2 \geq 2$. Hence, $s > 1$.
Thus, $\frac{n}{q} = b_1 - \frac{1}{b_2 - s}$, where $s > 1$ ..... (II)
The above steps can be repeated to get $s_1 > 1$ such that
$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - s_1}}$
Hence, the process does not end after finitely many steps. This is a contradiction, since, we have finitely many b_is.
Case (ii) If $b_1 < \left\lfloor \frac{n}{q} \right\rfloor + 1$.
In this case, we can write $\frac{n}{q} = b_1 + \frac{r}{q}$, for some $r > 0$.
Thus, $\frac{n}{q} = b_1 - \frac{1}{-q/r} = b_1 - \frac{1}{b_2 - s}$; where $b_2 - s = -\frac{q}{r}$, giving $s = b_2 + \frac{q}{r}$
Now, we have $r > 0$, $q > 0$ and $b_2 \geq 2$. Thus, $s \geq 2 > 1$
Hence, we have, $\frac{n}{q} = b_1 - \frac{1}{b_2 - s}$ with $s > 1$. Again as in case (i), the process does not end after finitely many steps.
Hence, we must have $b_1 = \left\lfloor \frac{n}{q} \right\rfloor + 1$.
Now by replacing n by q and q by r, we get b_2 is unique. Similarly we get that all b_is are unique. [5]

(c) To prove that if $b_j > 2$ for some $j \in \{1, 2, \ldots, m\}$ then
$\sum_{i=1}^{m} (b_i - 2) < 2(n - q - 1)$.

We apply second principle of induction on n.
Observe that for $n = 2,\ n = 3,\ n = 4$, there is no value of q satisfying all the conditions.
To prove for $n = 5$:
Case (i) $q = 2$. We have $\frac{5}{2} = 3 - \frac{1}{2}$ and by (I), such expression is unique.
Case (ii) $q = 3$. We have $\frac{5}{3} = 2 - \frac{1}{3}$ and by (I), such expression is unique.
Let $n > 5$. Assume the result is true for all $k \leq n$.
To prove the result for $n + 1$.
Let $1 < q < n + 1$ and $\gcd(n + 1, q) = 1$
$\frac{n+1}{q} = b_1 - \frac{1}{q/r} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ddots - \frac{1}{b_{m-1} - \frac{1}{b_m}}}}$
By uniqueness of b_is, we have, $\frac{q}{r} = b_2 - \frac{1}{b_3 - \ddots - \frac{1}{b_{m-1} - \frac{1}{b_m}}}$. If no $b_i > 2$ for $i = 2, 3, \ldots, m$ then we get, $b_i = 2$, for each $i = 2, 3, \ldots, m$. So, we must have $b_1 > 2$.
In this case, we get $\frac{n+1}{q} = b_1 - \frac{1}{2 - \frac{1}{2 - \ddots - \frac{1}{2 - \frac{1}{2}}}}$.
Observe that $\frac{1}{2 - \frac{k-1}{k}} = \frac{k}{k+1}$. Hence $\frac{n+1}{q} = b_1 - \frac{k-1}{k}$.
Now $n + 1 = b_1 q - \frac{(k-1)}{k} q$. Thus $k | q$. Let $q = km$. Then we have
$n + 1 = b_1 q - (k - 1)m = b_1 q - km + m = b_1 q - q + m$.
Therefore $n - q = b_1 q - 2q + m - 1 = q(b_1 - 2) + (m - 1)$.
Thus we get $2(n + 1 - q - 1) = 2(n - q) = 2q(b_1 - 2) + 2(m - 1) > b_1 - 2$ as required.

Now, if $b_i > 2$, for some $i = 2, 3, 4 \ldots, m$ then applying induction hypothesis to q, we get,
$\sum_{i=2}^{m} (b_i - 2) < 2(q - r - 1)$
We need to prove that $\sum_{i=1}^{m} (b_i - 2) < 2(n + 1 - q - 1)$.
It is enough to prove that $2(q - r - 1) + b_1 - 2 \leq 2(n + 1 - q - 1)$
That is to prove that $2q - 2r - 2 + b_1 - 2 \leq 2(b_1 q - r - q - 1)$,
which is equivalent to $b_1 - 2 \leq 2q(b_1 - 2)$ which holds. Hence, the result is true for $n + 1$. Thus by second principal of induction, the result is true for all $n \geq 5$. [5]

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 07/02/2021 Max. Marks: 50
Time: 12.00 noon to 1.30 p.m.
N.B.: Part I carries 20 marks, Part II carries 20 marks and Part III carries 10 marks.

Part I: MCQ with single correct answer
N.B. Each question in Part I carries 2 marks for correct answer and -1 mark for wrong answer.

1. For matrices X, Y define $[X, Y] = XY - YX$. For $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, rank $[X, Y]$
   (a) equals rank$X$
   (b) is strictly less that rank (XY)
   (c) is strictly greater than rank (XY)
   (d) equals 0.

   Ans:(c)

2. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has
   (a) at least one root in $(0, 1)$.
   (b) one root in $(1, 2)$ and other in $(-1, 0)$.
   (c) both imaginary roots.
   (d) a repeated root

   Ans:(a)

3. Let $f(x)$ be the function defined on $\mathbb{R}$ such that $f(x) = x$, for all $x \leq 1$ and $f(x) = ax^2 + bx + c$, for $x > 1$. The triples (a, b, c) such that $f(x)$ is differentiable at all real x are of the form:
   (a) $(a, 1 - 2a, a)$
   (b) $(1, 0, 0)$ only
   (c) $(a, -2a, a)$
   (d) $(0, 1, -1)$ only

   Ans:(a)

4. If $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, for all $x \in (0,\ 1)$, then $\int_0^x f(t)dt$ equals
   (a) $\tan^{-1}(x)$
   (b) $\sin^{-1}(x)$
   (c) $\sin(x)$
   (d) $\tan(x)$

Ans:(a)

5. The system of equations

$$2x + py + 6z = 8$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

has no solution for

(a) $p \neq 2, q \neq 3$
(b) $p \neq 2, q = 3$
(c) $p = 2, q = 3$
(d) $p = 2, q \neq 3$

Ans:(b)

6. Let f be any differentiable function on $\mathbb{R}$ with $f(-2) = 16, f(4) = 4$ and $f(8) = 24$. For which of the following three values of λ the equation $f'(x) = \lambda$ must have a solution?

(a) $0, 5, 10$
(b) $-5, 0.8, 3$
(c) $-2, 0.8, 5$
(d) $-2, 0, 14$

Ans:(c)

7. If $\int \frac{f(x)}{\log \sin x} dx = \log \log \sin x$, then $f(x) =$

(a) $\sin x$
(b) $\cos x$
(c) $\log \sin x$
(d) $\cot x$

Ans:(d)

8. Let $x_n = \{1, -1, 2, -2, 3, -3, \cdots\}$ and $y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}, n \geq 1$, then the sequence $\{y_n\}$ is

(a) monotonic but not convergent.
(b) convergent.
(c) bounded but not convergent.
(d) not bounded.

Ans:(c)

9. Let $A = \{(x, y) \in \mathbb{R}^2 \ : \ xy \in \mathbb{Z}\}$ and $B = \{(x, y) \in \mathbb{R}^2 \ : \ x + y \in \mathbb{Q}\}$. Then

(a) $A \cap B$ is countable and B is uncountable.
(b) $A \cap B$ is uncountable.
(c) A is countable.
(d) B is countable.

Ans: (a)

10. We have two glasses. One having milk and other having water in it, in exactly the same quantities. One spoon of milk is transferred to water and then one spoon of mixture is replaced in milk. Then which of the following is true?

    (a) The amount of milk in water is bigger than the amount of water in milk.

    (b) The amount of milk in water is smaller than the amount of water in milk.

    (c) The amount of milk in water is equal to the amount of water in milk.

    (d) The information is insufficient.

    Ans:(c)

Part II: Numerical Questions

N.B. The answer to each question in Part II is an integer. Each question in Part II carries 2 marks. No marks will be deducted for wrong answer.

1. For a complex number z, if $|z+4| \leq 3$, then the maximum value of $|z+1|$ is .....

   Ans: 6

2. The number of continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $(f(x))^2 = xf(x)$ is .....

   Ans: 4

3. For any positive integer n, let $S(n)$ be the sum of digits of n and $T(n) = S(n) + n$. The smallest value of n such that $T(n) = T(n+k)$ for some positive integer k is .....

   Ans: 91

4. $\lim\limits_{x\to 0} \dfrac{\cos(4x) - 1}{x^2} =$ .....

   Ans: -8

5. If $f(x) = x^3 + \sin x$, then the value of $\displaystyle\int_{-\pi/2}^{\pi/2} (x^2+1)(f(x) + f''(x))dx$ is .....

   Ans: 0

6. The **sum** of all real solutions of the equation $(x^2 - 5x + 5)^{(x^2+4x-60)} = 1$ is .....

   Ans: 3

7. Let $a_1 = \dfrac{10}{11}$ and $a_n = \dfrac{10}{11} + \dfrac{9}{11}a_{n-1}$ for $n \geq 2$, then $\lim\limits_{n\to\infty} a_n =$ .....

   Ans: 5

8. Three of the roots of a polynomial $p(X)$ with rational coefficients are: 2020, $\sqrt{7}$, $2021 - 2022i$. The least possible degree of $p(X)$ is .....

   Ans: 5

9. The greatest negative integer satisfying $x^2 - 4x - 77 < 0$ and $x^2 > 4$ is .....

   Ans: -3

10. The number of real values of a such that there exists a real number x satisfying the equation $a^2 - 2a\sin x + 1 = 0$ is

    Ans: 2

Part III: Multiple Select Questions

N.B. Each question in Part III carries 2 marks. No marks will be deducted for wrong answer. Each question may have more than one correct alternatives. A candidate gets 2 marks if he/she selects all the correct answers only and no wrong answers.

1. A four letter word is converted into a matrix form by writing its letters, say $ABCD$ as the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Each such matrix is then replaced by the corresponding letter in the alphabet via the rule $A \mapsto 1$, $B \mapsto 2, \ldots, Z \mapsto 26$. The resulting matrix is further reduced modulo 5. If the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ has been obtained by the above procedure, the following words are among the possible original words

   (a) GOOD
   (b) QEOS
   (c) GEOD
   (d) QEED

   Ans: (a),(b),(c),(d).

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all real numbers x, y. Then

   (a) f^{-1} may not exist
   (b) f is one-one
   (c) f is onto
   (d) f^{-1} is continuous

   Ans: (b),(c),(d).

3. Let $p(x) = x^2 + x + 2020$. If $x_n = p(n), y_n = x_{n+1} - x_n$ and $z_n = y_{n+1} - y_n$, then

   (a) The sequence $\{y_n\}$ is convergent.
   (b) The sequence $\{z_n\}$ converges to zero.
   (c) The sequence $\{z_n\}$ is constant.
   (d) The sequence $\{y_n\}$ is monotonic.

   Ans: (c),(d).

4. Let $x = 0.a_1a_2a_3a_4\cdots$ be the decimal expansion of $x \in (0, 1)$, then x is rational if

   (a) $a_n = 0$ if $n = k^2$ and $a_n = 1$ if $n \neq k^2$ for any positive integer k.
   (b) $a_n = 0$ if n is odd and $a_n = 1$ if n is even.
   (c) $a_n = 0$ if $n = k!$ and $a_n = 1$ if $n \neq k!$ for any positive integer k.
   (d) $a_n = 0$ if $n \leq 2020$ and $a_n = 1$ if $n \geq 2021$.

   Ans: (b),(d).

5. If $1, w_1, w_2, w_3, w_4, w_5$ are distinct roots of $x^6 - 1$, then

   (a) $1 + w_i + w_i^2 + w_i^3 + w_i^4 + w_i^5 = 0$ for $i = 1, 2, 3, 4, 5$.
   (b) $1 + w_i^2 + w_i^4 + w_i^6 + w_i^8 + w_i^{10} = 0$ for $i = 1, 2, 3, 4, 5$.
   (c) $1 + w_i^3 + w_i^6 + w_i^9 + w_i^{12} + w_i^{15} = 0$ for $i = 1, 2, 3, 4, 5$.
   (d) $1 + w_i^5 + w_i^{10} + w_i^{15} + w_i^{20} + w_i^{25} = 0$ for $i = 1, 2, 3, 4, 5$.

   Ans: (a),(d).

MADHAVA MATHEMATICS COMPETITION (Second Round)
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 15/02/2021 Max. Marks: 50
Time: 12.00 noon to 2.00 p.m.
N.B.: Part I carries 30 marks and Part II carries 20 marks.

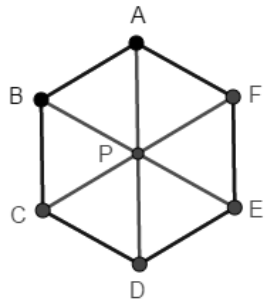
Part I

N.B. Each question in Part I carries 6 marks.

1. Let $p(x) = x^4 - ax^3 + bx^2 - cx + 480$ be a polynomial whose all zeros are integers greater than 1. Let further, $|p(4)| + |p'(4)| = 0$. Find all such polynomials. Also find the least and greatest possible value of a.

2. Let $M = \begin{pmatrix} 1/4 & 1/8 & 1/16 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$. Find the $\lim_{n\to\infty} M^n \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

3. In the diagram given below, the points A, B, C, D, E, F and P represent cities and edges joining them represent roads connecting the cities.



Two cities are said to be adjacent if there is a single edge joining them. (For example, the cities B, C are adjacent, but B, E are not). A tourist moves to an adjacent city on each day. Starting with P, the tour ends if the tourist happens to come back at P.

(a) Find the probability that the tour ends on the second day.

(b) Find the probability that the tour ends on the third day.

(c) For any natural number n, find the probability that the tour ends on n^{th} day.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that $f(c) = 0$ for some $c > 0$. Let $b > 0$ and $x_0 > c$.

   (a) Consider a line L_0 joining $(0, b)$ and $(x_0, 0)$. Prove that there exists a unique real number x_1 such that $(x_1, f(x_1))$ lies on L_0.

   (b) Let L_1 be a line joining $(0, b)$ and $(x_1, 0)$. Define the sequence $\{x_n\}$ inductively applying the above procedure. Prove that the sequence $\{x_n\}$ is convergent. Further, show that $\{x_n\}$ converges to c.

5. Prove that every integer between 1 and $n!$, $(n \geq 3)$ can be expressed as the sum of at most n distinct divisors of $n!$.

Part II

N.B. Each question in Part II carries 10 marks.

1. (a) Show that there does not exist a function $f : (0, \infty) \to (0, \infty)$ such that $f''(x) \leq 0$ for all x and $f'(x_0) < 0$ for some x_0.

   (b) Let $k \geq 2$ be any integer. Show that there does not exist an infinitely differentiable function $f : (0, \infty) \to (0, \infty)$ such that $f^{(k)}(x) \leq 0$ for all x and $f^{(k-1)}(x_0) < 0$ for some x_0. Here, $f^{(k)}$ denotes the k^{th} derivative of f.

2. Prove that every monic polynomial $f(x)$ of degree n over $\mathbb{R}$ can be expressed as an arithmetic mean of two monic polynomials of degree n over $\mathbb{R}$ each having n real roots.

MADHAVA MATHEMATICS COMPETITION 2021 (Second Round)
Solutions and scheme of marking

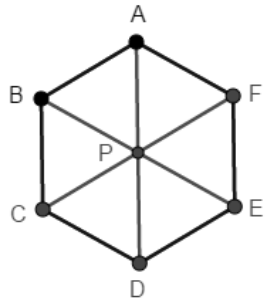
Part I

N.B. Each question in Part I carries 6 marks.

1. Let $p(x) = x^4 - ax^3 + bx^2 - cx + 480$ be a polynomial whose all zeros are integers greater than 1. Let further, $|p(4)| + |p'(4)| = 0$. Find all such polynomials. Also find the least and greatest possible value of a.
   Solution: The condition $|p(4)| + |p'(4)| = 0$ implies that $p(4) = 0$ and $p'(4) = 0$. Therefore 4 is a repeated root of $p(x)$. [2]
   Suppose the roots of the polynomial $p(x)$ are $4, 4, \alpha, \beta$. Thus, $\alpha\beta = 30$. Now α, β being integers greater than 1, the only possible values of α, β are $3, 10$ or $5, 6$ or $2, 15$. [2]
   It is now clear that the least possible value of a is $5+6+8 = 19$ and the greatest possible value of a is $15 + 2 + 8 = 25$. [1]
   There are three polynomials satisfying the given conditions:
   $p_1(x) = (x-4)^2(x-3)(x-10), p_2(x) = (x-4)^2(x-5)(x-6)$,
   $p_3(x) = (x-4)^2(x-2)(x-15)$. [1]

2. Let $M = \begin{pmatrix} 1/4 & 1/8 & 1/16 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$. Find the $\lim_{n\to\infty} M^n \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$
   Solution: Observe that $X = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ [2]
   Further, it can be seen by induction that $M^n \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ and
   $M^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4^n \\ 0 \\ 0 \end{pmatrix}$ for all $n \in \mathbb{N}$. [3]
   Thus, $\lim_{n\to\infty} M^n \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \lim_{n\to\infty} M^n \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \lim_{n\to\infty} M^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$. [1]

3. In the diagram given below, the points A, B, C, D, E, F and P represent cities and edges joining them represent roads connecting the cities.



Two cities are said to be adjacent if there is a single edge joining them. (For example, the cities B, C are adjacent, but B, E are not). A tourist moves to an adjacent city on each day. Starting with P, the tour ends if the tourist happens to come back at P.

(a) Find the probability that the tour ends on the second day.

(b) Find the probability that the tour ends on the third day.

(c) For any natural number n, find the probability that the tour ends on n^{th} day.

Solution: Let P_n denote the probability that the tour ends on the n^{th} day. Clearly, the value of P_1 is 0. (a) It is clear that $P_2 = 1/3$, as no matter where the tourist goes on day 1, from that vertex (any of A, B, C, D, E, F) out of three options, there is exactly one option to come back to P. [1]
(b) We first note that in this case, tourist does not reach P on day 1 or day 2. By symmetry, without loss we may assume that on day 1, the tourist reaches one of the cities A, B, C, D, E, F with probability 1.
Suppose the tourist is at A at the end of day 1. Out of three options at A, the tourist has two options B or F in order not to reach P on day 2. This event has probability 2/3. Again by symmetry, we may assume that the tourist is at B at the end of day 2. Out of three options at B, there is exactly one option to come back to P. This event has probability 1/3. By multiplication theorem, required probability is $P_3 = 1 \times 2/3 \times 1/3 = 2/9$. [3]
(c) The argument for the general case is similar to that in part (b). Indeed, on day 2, day 3, $\cdots$, day $(n-1)$ the tourist has two options out of three in order not to reach P. Each of these events has probability 2/3 and again as in part (b) on n^{th} day the event has probability 1/3.
Therefore $P_n = 1 \times 2/3 \times 2/3 \times \cdots \times 2/3 \times 1/3 = 2^{n-2}/3^{n-1}$. [2]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that $f(c) = 0$ for some $c > 0$. Let $b > 0$ and $x_0 > c$.

(a) Consider a line L_0 joining $(0, b)$ and $(x_0, 0)$. Prove that there exists a unique real number x_1 such that $(x_1, f(x_1))$ lies on L_0.

(b) Let L_1 be a line joining $(0, b)$ and $(x_1, 0)$. Define the sequence $\{x_n\}$ inductively applying the above procedure. Prove that the sequence $\{x_n\}$ is convergent. Further, show that $\{x_n\}$ converges to c.

Solution: Define a function $g_0 : \mathbb{R} \to \mathbb{R}$ by $g_0(x) = x_0 - \dfrac{x_0}{b} f(x) - x$. As f is continuous so is g_0. Further, $g_0(c) = x_0 - c > 0$ and $g_0(x_0) = -\dfrac{x_0}{b} f(x_0) < 0$.
By Intermediate Value Theorem, there exist $x_1 \in (c, x_0)$ such that $g_0(x_1) = 0$ and hence $x_1 = x_0 - \dfrac{x_0}{b} f(x_1)$. As f is strictly increasing function, we have unique such x_1. Since the equation of the line L_0 is $y = b - \dfrac{b}{x_0} x$, the point $(x_1, f(x_1))$ lies on the line L_0.
Now define the function $g_1 : \mathbb{R} \to \mathbb{R}$ by $g_1(x) = x_1 - \dfrac{x_1}{b} f(x) - x$. Using the similar argument as above, we get the unique point $(x_2, f(x_2))$ which lies on the line L_1.
Applying the above procedure repeatedly, we get a sequence (x_n) such that

$x_{n+1} = x_n - \frac{x_n}{b} f(x_{n+1})$ and $x_{n+1} < x_n$, $\forall n \in \mathbb{N}$. [3]

Thus the sequence (x_n) is decreasing and bounded below by c and hence it is a convergent sequence.

Let $p = \lim_{n\to\infty} x_n$. Then, using the relation, $x_{n+1} = x_n - \frac{x_n}{b} f(x_{n+1})$, we get, $f(p) = 0$.

But, f is strictly increasing and $f(c) = 0$, hence we must have $c = p$. Thus, (x_n) converges to c. [3]

5. Prove that every integer between 1 and $n!$, $(n \geq 3)$ can be expressed as the sum of at most n distinct divisors of $n!$.

Solution: It can be seen that the result is true for $n = 3$.

Assume that the result is true for $n - 1$ for $n \geq 4$. Consider any k, $1 < k < n!$.

Let $k = nq + r$ where $0 \leq r < n$. [3]

Observe that $0 < q \leq \frac{k}{n} < \frac{n!}{n} = (n-1)!$.

By induction hypothesis, there exists $d_1, d_2, \cdots, d_m$ such that $q = d_1 + d_2 + \cdots + d_m$ where each d_i divides $(n-1)!$ and $m \leq n-1$.

We may assume $1 \leq d_1 < d_2 < \cdots < d_m$.

If $r = 0$, then $k = nq = nd_1 + nd_2 + \cdots + nd_m$ where each nd_i divides $n!$, nd_i are all distinct and $m \leq n - 1 < n$.

If $r \neq 0$, then $1 \leq r \leq n - 1 < n$. This implies r divides $n!$ and $r < n < nd_1$.

Hence $k = nq + r = r + nd_1 + nd_2 + \cdots + nd_m$, where $m + 1 \leq n$, as required.

Thus the result is true for n and hence for every natural number. [3]

Part II

N.B. Each question in Part II carries 10 marks.

1. (a) Show that there does not exist a function $f : (0, \infty) \to (0, \infty)$ such that $f''(x) \leq 0$ for all x and $f'(x_0) < 0$ for some x_0.

   (b) Let $k \geq 2$ be any integer. Show that there does not exist an infinitely differentiable function $f : (0, \infty) \to (0, \infty)$ such that $f^{(k)}(x) \leq 0$ for all x and $f^{(k-1)}(x_0) < 0$ for some x_0. Here, $f^{(k)}$ denotes the k^{th} derivative of f.

First Solution:

(a) Suppose there exists a function $f : (0, \infty) \to (0, \infty)$ such that $f''(x) \leq 0$ for all x and $f'(x_0) < 0$ for some x_0. Since f' is a decreasing function and $f'(x_0) < 0$, we get $f'(x) < 0$ for all $x > x_0$.

By Fundamental Theorem of Calculus, we have $f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt$.

Thus there exists $x_1 > x_0$ such that $f(x_1) < 0$, a contradiction. [6]

(b) Suppose there exists an infinitely differentiable function $f : (0, \infty) \to (0, \infty)$ such that $f^{(k)}(x) \leq 0$ for all x and $f^{(k-1)}(x_0) < 0$ for some x_0. Since $f^{(k-1)}$ is a decreasing function and $f^{(k-1)}(x_0) < 0$, we get $f^{(k-1)}(x) < 0$ for all $x > x_0$.

By Fundamental Theorem of Calculus, we have $f^{(k-2)}(x) = f^{(k-2)}(x_0) + \int_{x_0}^{x} f^{(k-1)}(t)dt$.

Thus there exists $x_1 > x_0$ such that $f^{(k-2)}(x) < 0$ for $x > x_1$. By a repeated application of the Fundamental Theorem of Calculus, we get that, $f(x) < 0$ for some real number x, a contradiction. [4]

Second Solution: Suppose there exists a function $f : (0, \infty) \to (0, \infty)$ such that $f''(x) \leq 0$ for all x and $f'(x_0) < 0$ for some x_0. Thus f' is a decreasing function. Define $g(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$.

Note that $g(x_0) = 0$ and $g'(x) = f'(x) - f'(x_0) < 0$ for all $x > x_0$.
Since g is a decreasing function on (x_0, ∞), we get $g(x) \leq 0$ for all $x > x_0$.
Now $f'(x_0) < 0$ implies, there exists $x_1 > x_0$ such that $f(x_1) < 0$. [6]
(b) For the general $k \geq 2$, proof can be given by similar argument. [4]

2. Prove that every monic polynomial $f(x)$ of degree n over $\mathbb{R}$ can be expressed as an arithmetic mean of two monic polynomials of degree n over $\mathbb{R}$ each having n real roots.
Solution:
Case 1: If $degf(x) = 1$ then $f(x) = x + a$ for some $a \in \mathbb{R}$. Take $P(x) = x$ and $Q(x) = x + 2a$. Then, both $P(x)$ and $Q(x)$ are of degree 1, having real root and $f(x) = \dfrac{P(x) + Q(x)}{2}$. [1]
Assume deg $f(x) > 1$.
Let $Q(x) = x^n - k(x-2)(x-4)\ldots(x-2(n-1))$ and $P(x) = 2f(x) - Q(x)$, where $k > 0$ is chosen sufficiently large so that $k > x^n$ and $P(x)$ and $Q(x)$ have opposite signs for all $x \in [0, 2n]$. Note that the choice of $P(x)$ is made in such a way that if $Q(x)$ has n real roots then $P(x)$ also will have n real roots.
Case 2: Let deg $f(x) = 2$
Here, $Q(x) = x^2 - k(x-2)$. Observe that $Q(2) > 0$ and $Q(3) < 0$. Thus, $Q(x)$ has root in the interval $(2, 3)$ and consequently, will have two real roots. [2]
Case 3: Let deg $f(x) = 3$
Here, $Q(x) = x^3 - k(x-2)(x-4)$. Observe that $Q(0) < 0$ and $Q(2) > 0$ and $Q(4) > 0, Q(6) < 0$. Thus $Q(x)$ has a root in each of the intervals $(0, 2), (4, 6)$. Consequently $Q(x)$ has 3 real roots. [2]
Case 4: Suppose n is even and $n \geq 4$.
Observe that $Q(2) > 0$, $Q(4) > 0$, $\ldots$, $Q(2n-2) > 0$ and $Q(1) > 0$, $Q(3) < 0$, $Q(5) > 0$, $Q(7) < 0$, $\ldots$, $Q(2n-3) > 0$, $Q(2n-1) < 0$ and $Q(x) > 0$ for sufficiently large $x > 2n-2$.
Therefore $Q(x)$ has two roots in each of the intervals $(2, 4)$, $(6, 8)$, $\ldots$, $(2n-6, 2n-4)$ and one real root in the interval $(2n-2, 2n)$. Therefore there are two roots in each of $(n-2)/2$ intervals and one root in $(2n-2, 2n)$ resulting into $n-1$ real roots. Consequently $Q(x)$ has n real roots.
Case 5: Suppose n is odd and $n \geq 5$.
Observe that $Q(2) > 0$, $Q(4) > 0$, $\ldots$, $Q(2n-2) > 0$ and $Q(0) < 0$, $Q(5) < 0$, $Q(9) < 0$, $\ldots$, $Q(2n-5) < 0$ and $Q(x) < 0$ for sufficiently large $x > 2n-2$.
Therefore $Q(x)$ has two roots in each of the intervals $(4, 6)$, $(8, 10)$, $\ldots$, $(2n-6, 2n-4)$ and one real root in each of the intervals $(0, 2)$ and $(2n-2, 2n)$. Therefore there are two roots in each of $(n-3)/2$ intervals and one each in $(0, 2)$ and $(2n-2, 2n)$ resulting into $n-1$ real roots. Consequently $Q(x)$ has n real roots. [5]
(Note: 5 marks for both cases and 3 marks for only one case)

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 23/01/2022 Max. Marks: 50
Time: 12.00 noon to 1.30 p.m.
N.B.: Part I carries 20 marks, Part II carries 20 marks and Part III carries 10 marks.

Part I: MCQ with single correct answer
N.B. Each question in Part I carries 2 marks for correct the answer and -1 mark for a wrong answer.

1. Let the sequence $\{x_n\}$ be defined as follows: $x_1 = 1$ and x_n is the smallest prime factor of n. Then the sequence $\{x_n\}$
   (a) is monotonic
   (b) diverges to infinity
   (c) has a convergent subsequence
   (d) is not bounded below

   Ans:(c)

2. The equation $x^6 - x - 1 = 0$ has
   (a) no positive real root
   (b) exactly one positive real root
   (c) exactly two positive real roots
   (d) all roots are real and positive

   Ans:(b)

3. The value of θ $(0 \leq \theta \leq \pi/2)$ for which the number $\dfrac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary is
   (a) $\pi/6$
   (b) $\pi/3$
   (c) $sin^{-1}(\sqrt{3}/4)$
   (d) $sin^{-1}(1/\sqrt{3})$

   Ans:(d)

4. Consider the curve $y = 2x^4 + 7x^3 + 3x - 5$. Let $P_i = (x_i, y_i)$ be four distinct points of intersection of a line with the given curve. Then the value of $\dfrac{x_1 + x_2 + x_3 + x_4}{4}$ is
   (a) $-7/8$
   (b) $-7/2$
   (c) $7/8$
   (d) $7/2$

   Ans:(a)

5. If one root of the equation $x^2 + px + 12 = 0$ is 4 while the equation $x^2 + px + q = 0$ has equal roots, the value of q is

   (a) $4/49$
   (b) $49/4$
   (c) $-49/4$
   (d) $-4/49$

   Ans:(b)

6. Which of the following equations has greatest number of real solutions?

   (a) $x^3 = 10 - x$
   (b) $x^2 + 5x - 7 = x + 8$
   (c) $7x + 5 = 1 - 3x$
   (d) $e^x = x$

   Ans:(b)

7. Let $\gcd(a, b) = 1$, then $\gcd(a + b, a^2 - ab + b^2) =$

   (a) 2
   (b) 1 or 2
   (c) 1 or 3
   (d) 2 or 3

   Ans:(c)

8. Suppose f is continuous in $[0, 2]$ and differentiable in $(0, 2)$. If $f(0) = 0$ and $|f'(x)| \leq 1/2$ for all $x \in [0, 2]$, then

   (a) $|f(x)| \leq 1$
   (b) $|f(x)| \leq 1/2$
   (c) $f(x) = 2x$
   (d) $f(x) = 3$ for at least one $x \in [0, 2]$.

   Ans:(a)

9. Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. The number of functions from A to B such that exactly one element in B has two pre-images is

   (a) 12
   (b) 18
   (c) 24
   (d) 36

   Ans:(d)

10. Consider a square matrix $A = [a_{ij}]$ of order 3, all whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0. Also $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq 3$. Then the number of such matrices is

    (a) 12

    (b) 9

    (c) 3

    (d) 1

    Ans:(a)

Part II: Numerical Questions

N.B. The answer to each question in Part II is an integer. Each question in Part II carries 2 marks. No marks will be deducted for wrong answer.

1. If the matrix $A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$ satisfies the equation $A^2 - kA + 2I = 0$, then the value of k is .....

   Ans: 1

2. The remainder when $\sum_{r=1}^{100} r!$ is divided by 12 is .....

   Ans: 9

3. $3^1 \times 3^{1/2} \times 3^{1/4} \times 3^{1/8} \times \cdots =$

   Ans: 9

4. Let f be a differentiable real valued function on $(-1, 4)$ such that $f(3) = 5$ and $f'(x) \geq -1$ for all x. Then the greatest possible value of $f(0)$ is .....

   Ans: 8

5. Suppose the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$ is $ax^2 + bx + c$. Then the value of b is .....

   Ans: 5

6. If the sum of the series $a + ar + ar^2 + \cdots$ is 4 and the sum of the series $a^3 + a^3r^3 + a^3r^6 + \cdots$ is 192. Then the value of a is .....

   Ans: 6

7. The sum of the series $1 + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \cdots + \dfrac{1}{1+2+\cdots+n} + \cdots$ is .....

   Ans: 2

8. The number of ways to write $5 = a_1 + a_2 + \cdots + a_k$, where all a_i are integers satisfying $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$ is .....

   Ans: 5

9. The number of solutions of $\sin^5 x + \cos^5 x = 1$ in $[0, \pi]$ is .....

   Ans: 2

10. If $A = [a_{ij}]$ is a square matrix of order 5 such that the entry $a_{ij} = 1$ if and only if $i = j$ or $i + j = 6$, and 0 otherwise, then the rank of A is .....

    Ans: 3

Part III: Multiple Select Questions

N.B. Each question in Part III carries 2 marks. No marks will be deducted for wrong answer. Each question may have more than one correct alternatives. A candidate gets 2 marks if he/she selects all the correct answers only and no wrong answers.

1. If the equation $ax^2 + bx + c = 0, (a > 0)$ has two roots α, β such that $\alpha < -2$ and $\beta > 2$, then

   (a) $b^2 - 4ac > 0$

   (b) $c < 0$

   (c) $a + |b| + c < 0$

   (d) $4a + 2|b| + c < 0$

   Ans: (a),(b),(c),(d).

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $a \in \mathbb{R}$. Define $g : [a, \infty) \to \mathbb{R}$ as $g(x) = \sup\{f(t) : t \in [a, x]\}$. Then

   (a) g is continuous

   (b) g is monotonically decreasing

   (c) g is monotonically increasing

   (d) g is differentiable whenever f is differentiable.

   Ans: (a),(c).

3. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non-zero real numbers. We say that $\{a_n\}$ and $\{b_n\}$ are almost equal if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Which of the following sequences are almost equal?

   (a) $a_n = n + \sqrt{n}, b_n = n$.

   (b) $a_n = n^2 + \sqrt{n}, b_n = n$.

   (c) $a_n = n!, b_n = n^n$.

   (d) $a_n = (1 + \frac{1}{n})^n, b_n = e$.

   Ans: (a),(d).

4. For which of the following functions $f : \mathbb{R} \to \mathbb{R}$ the ratio $\dfrac{f(k) - f(m)}{k - m}$ is constant for all $k, m \ \ (k \neq m)$?

   (a) $f(x) = x^2 + x$
   (b) $f(x) = x + |x|$
   (c) $f(x) = 4x + 7$
   (d) $f(x) = |x|$

   Ans: (c).

5. Let α be a 2022^{nd} root of unity. Then which of the following are possible values of $1 + \alpha + \alpha^2 + \cdots + \alpha^{2021}$?

   (a) 0
   (b) i
   (c) 2021
   (d) 2022

   Ans:(a),(d)

MADHAVA MATHEMATICS COMPETITION (Second Round)
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 03/03/2022 Max. Marks: 50
Time: 12.00 noon to 2.00 p.m.
N.B.: Part I carries 30 marks and Part II carries 20 marks.

Part I

N.B. Each question in Part I carries 6 marks.

1. Let the positive integers a, b, c be such that $a \geq b \geq c$ and $(a^x - b^x - c^x)(x-2) > 0$ for all $x \neq 2$. Show that a, b, c are sides of a right angled triangle.
2. Find all real numbers x, y such that the fractional part of $\dfrac{x+4y+1}{x^2+y^2+19}$ is $\dfrac{1}{2}$.
3. Let f be a quadratic polynomial. Show that there exist quadratic polynomials g, h such that $f(x)f(x+1) = g(h(x))$.
4. Determine the number of all $m \times n$ matrices with entries 0 or 1 such that the number of 1's in each row and the number of 1's in each column are all even.
5. Find all non-negative integer solutions to the system of equations

$$3x^2 - 2y^2 - 4z^2 + 54 = 0$$
$$5x^2 - 3y^2 - 7z^2 + 74 = 0$$

Part II

N.B. Each question in Part II carries 10 marks.

1. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that f' is continuous and $f(0) = 0, f(1) = 1$.
   (a) Show that there exists x_1 in $(0,1)$ such that $\dfrac{1}{f'(x_1)} = 1$. [1]
   (b) Show that there exist distinct x_1, x_2 in $(0,1)$ such that $\dfrac{1}{f'(x_1)} + \dfrac{1}{f'(x_2)} = 2$. [4]
   (c) Show that for a positive integer n, there exist distinct $x_1, x_2, \cdots, x_n$ in $(0,1)$ such that $\displaystyle\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n$. [5]
2. Let $\mathcal{P}_n$ denote the collection of polynomials of degree n such that the polynomial and all its derivatives have integer roots.
   (a) Find a polynomial in $\mathcal{P}_2$ having at least two distinct roots. [2]
   (b) Find a polynomial in $\mathcal{P}_3$ having at least two distinct roots. [3]
   (c) For any polynomial P in $\mathcal{P}_n$, show that the arithmetic mean of all roots of P is also an integer. [5]

MADHAVA MATHEMATICS COMPETITION (Second Round)
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: /02/2022 Max. Marks: 50
Time: 12.00 noon to 2.00 p.m.
N.B.: Part I carries 30 marks and Part II carries 20 marks.

Part I

N.B. Each question in Part I carries 6 marks.

1. Let the positive integers a, b, c be such that $a \geq b \geq c$ and $(a^x - b^x - c^x)(x-2) > 0$ for all $x \neq 2$. Show that a, b, c are sides of a right angled triangle.
Solution: Let $f(x) = (a^x - b^x - c^x)$ where $x \in \mathbb{R}$.
For $x < 2$ implies $f(x) < 0$
For $x > 2$ implies $f(x) > 0$
By continuity of $f(x)$ we have
$\lim_{x \to 2} f(x) = f(2) = 0$
Thus $a^2 = b^2 + c^2$

2. Find all real numbers x, y such that the fractional part of $\dfrac{x+4y+1}{x^2+y^2+19}$ is $\dfrac{1}{2}$.

Solution: Note that since $x^2 + y^2 + 19 > 0$ and $\{\dfrac{x+4y+1}{x^2+y^2+19}\}$ is $\dfrac{1}{2}$, we have $x + 4y + 1 > 0$.

We can prove that $(x + 4y + 1) < (x^2 + y^2 + 19)$. Since $x^2 - x + y^2 - 4y + 18 = (x - 1/2)^2 + (y - 4)^2 + 7/4 > 0$, we have

$$\{\frac{x+4y+1}{x^2+y^2+19}\} = \frac{x+4y+1}{x^2+y^2+19} = \frac{1}{2}$$

Thus $2x + 8y + 2 = x^2 + y^2 + 19$
i.e. $(x-1)^2 + (y-4)^2 = 0$. We get $(x, y) = (1, 4)$.

3. Let f be a quadratic polynomial. Show that there exist quadratic polynomials g, h such that $f(x)f(x+1) = g(h(x))$. **Solution:**
$f(x) = a(x - \alpha)(x - \beta)$
$f(x+1) = a(x - \alpha + 1)(x - \beta + 1)$
$f(x) * f(x+1) = a^2(x^2 - x(\alpha + \beta - 1) + \alpha\beta - \alpha) * (x^2 - x(\alpha + \beta - 1) + \alpha\beta - \beta)$
Define $g(x) = a^2(x - \alpha)(x - \beta)$
$h(x) = x^2 - x(\alpha + \beta - 1) + \alpha\beta$

4. Determine the number of all $m \times n$ matrices with entries 0 or 1 such that the number of 1's in each row and the number of 1's in each column are all even.
Solution:
Fill $m - 1$ by $n - 1$ submatrix in any way. The number of ways to do this is $2^{(m-1)(n-1)}$. Since only 0 and 1 is allowed as entry and we have to just take care of parity, the remaining entries in the last row and column are uniquely forced. Thus the number of required $m \times n$ matrices is $2^{(m-1)(n-1)}$.

5. Find all non-negative integer solutions to the system of equations

$$3x^2 - 2y^2 - 4z^2 + 54 = 0$$
$$5x^2 - 3y^2 - 7z^2 + 74 = 0$$

Solution: Multiplying first equation by 5 and second by 3 we get

$$15x^2 - 10y^2 - 20z^2 + 5 * 54 = 0$$
$$15x^2 - 9y^2 - 21z^2 + 74 * 3 = 0$$

Substracting the second equation from the first, we get

$$z^2 - y^2 = -48$$
$$(y - z)(y + z) = 48$$

Trying the pairs 2×24, 6×8, 4×12
$y - z = 2, y + z = 24$, gives $y = 13, z = 11$ gives $x = 16$
$y - z = 6, y + z = 8$, gives $y = 7, z = 1$ gives $x = 4$
$y - z = 4, y + z = 12$ gives $y = 8, z = 4$ does not give a valid x.

Part II

N.B. Each question in Part II carries 10 marks.

1. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that f' is continuous and $f(0) = 0, f(1) = 1$.

   (a) Show that there exists x_1 in $(0,1)$ such that $\dfrac{1}{f'(x_1)} = 1$. [1]

   (b) Show that there exist distinct x_1, x_2 in $(0,1)$ such that $\dfrac{1}{f'(x_1)} + \dfrac{1}{f'(x_2)} = 2$. [4]

   (c) Show that for a positive integer n, there exist distinct $x_1, x_2, \cdots, x_n$ in $(0,1)$ such that $\displaystyle\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n$. [5]

Solution:

Since $f(0) = 0, f(1) = 1$ we have $\dfrac{f(1) - f(0)}{1 - 0} = f'(c)$

Therefore result is true for $n = 1$.

If $f'(x) \geq 1 \forall x$, then

$\dfrac{f(x)}{x} = f'(c_1) \geq 1$

Therefore $f(x) \geq x$.

If $f(x_0) > x_0$, then $1 - f(x_0) < 1 - x_0$

$f'(c_2) = \dfrac{1 - f(x_0)}{1 - x_0} < 1$ which is a contradiction. Thus $f(x) = x$.

There exists $a, b \in \mathbb{R}$ such that $f'(a) < 1$ and $f'(b) > 1$

Therefore $[1 - \epsilon, 1 + \epsilon] \subseteq Ran(f')$. Therefore we can find x_1, such that

$f'(x_1) = 1 - \epsilon/2$ i.e. $\dfrac{1}{f'(x_1)} = \dfrac{2}{2 - \epsilon}$.

Now we want to find x_2 such that $f'(x_2)$ satisfies

$\dfrac{1}{f'(x_2)} = 2 - \dfrac{1}{f'(x_1)} = 2 - \dfrac{2}{2 - \epsilon} = \dfrac{2 - 2\epsilon}{2 - \epsilon}$ i.e. $f'(x_2) = \frac{2-\epsilon}{2-2\epsilon}$. This is definitely greater than 1.

If $\dfrac{2 - \epsilon}{2 - 2\epsilon} < 1 + \epsilon$; then we can find such x_2.

Now $(1 + \epsilon)(2 - 2\epsilon) = 2 - 2\epsilon^2$.

Assuming without loss of generality that $\epsilon < 1$;

$2 - \epsilon < 2 - 2\epsilon^2$

Therefore $1 < \dfrac{2 - \epsilon}{2 - 2\epsilon} < 1 + \epsilon$

i.e. $\dfrac{2 - \epsilon}{2 - 2\epsilon} \in \text{Range}(f')$

Therefore it is possible to select x_2 such that

$\dfrac{1}{f'(x_1)} + \dfrac{1}{f'(x_2)} = 2$

Now $\dfrac{1}{f'(x_1)} < 1 < \dfrac{1}{f'(x_2)}$

Therefore $\exists\ \epsilon'$ such that

$\frac{1}{f'(x_1)} < 1-\epsilon' \quad 1+\epsilon' < \frac{1}{f'(x_2)}$,
i.e. in particular $[1-\epsilon', 1+\epsilon'] \subseteq \text{Range}(f)$
By similar argument $\exists x_3$ and x_4 such that $\frac{1}{f'(x_3)} + \frac{1}{f'(x_4)} = 2$

Therefore $\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} + \frac{1}{f'(x_3)} + \frac{1}{f'(x_4)} = 4$ and by contradiction x_1, x_2, x_3, x_4 are distinct. We can continue this process and get $x_1, x_2, x_3, \ldots, x_{2n}$ all distinct such that $\sum_{i=1}^{2k} \frac{1}{f'(x_i)} = 2k$. We have already seen that there exists $c = x$ such that $f'(x) = 1$. Since $f'(x_i) \neq 1 \; \forall \; i = 1, 2, \ldots, 2k$; this c is different from all x_i. Thus $\exists$ 2k+1 points such that $\sum_{i=1}^{2k+1} \frac{1}{f'(x_i)} = 2k+1$.

2. Let $\mathcal{P}_n$ denote the collection of polynomials of degree n such that the polynomial and all its derivatives have integer roots.

   (a) Find a polynomial in $\mathcal{P}_2$ having at least two distinct roots. [2]

   (b) Find a polynomial in $\mathcal{P}_3$ having at least two distinct roots. [3]

   (c) For any polynomial P in $\mathcal{P}_n$, show that the arithmetic mean of all roots of P is also an integer. [5]

Solution:
i) Let $f = (x-\alpha)(x-\beta)$; then $f' = (x-\alpha) + (x-\beta) = 2x - (\alpha+\beta)$
Thus the condition is $\alpha + \beta$ must be even.
ii) Let $f = (x-\alpha)(x-\beta)(x-\gamma)$
Then $f' = (x-\alpha)(x-\beta) + (x-\alpha)(x-\gamma) + (x-\beta)(x-\gamma)$
Let $f = (x-\alpha)^2(x-\gamma)$. Then $f' = (x-\alpha)^2 + 2(x-\alpha)(x-\beta)$
$f'' = 2(x-\alpha) + 2(x-\alpha) + 2(x-\beta) = 4(x-\alpha) + 2(x-\beta)$
$= 6x - 4\alpha - 2\beta$
$f'' = 0$ if $x = \frac{4\alpha + 2\beta}{6}$
So $6|4\alpha + 2\beta$. Put $\beta = 1 and \alpha = 4$. We can take $f(x) = (x-4)^2(x-1)$.
iii) Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in P_n$. Let the roots of $p(x)$ be $\alpha_1, \alpha_2, \ldots, \alpha_n$
Now $p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1$
$p''(x) = n(n-1)a_n x^{n-2} + (n-1)(n-2)a_{n-1}x^{n-3} + \ldots + 2a_2$ and so on.
$p^{(n-1)}(x) = n!a_n x + (n-1)!a_{n-1}$
Since $p^{(n-1)}(x)$ has integer roots, it is equal to $\frac{-(n-1)!a_{n-1}}{n!a_n}$
$\frac{\sum_{i=1}^n \alpha_i}{n} = \frac{-(n-1)a_{n-1}}{na_n}$ is thus an integer.

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Department of Mathematics, S. P. College, Pune (Autonomous)
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 29/01/2023 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The number of positive divisors of $2^{24} - 1$ is
   (A) 192 (B) 48 (C) 96 (D) 24.

2. The equation Re $(z^2) = 0$ represents
   (A) a circle (B) a pair of straight lines (C) an ellipse (D) a parabola.

3. If $A = \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$ and $\det A^3 = 125$, then the values of α are
   (A) ± 1 (B) ± 2 (C) ± 3 (D) ± 5.

4. Let A, B, C be three non-collinear points in a plane. The number of points at a distance 1 from A, 2 from B and 3 from C is
   (A) exactly 1 (B) at most 1 (C) at most 2 (D) always 0.

5. Let $A = \{x \in [-2, 3] : \cos x > 0\}$. Then
   (A) $\inf A = 0$ (B) $\sup A = \pi$ (C) $\inf A = -\pi/2$ (D) $\sup A = 3$.

6. Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| \leq \dfrac{2023}{n}|a_n - a_{n-1}|, \ \forall n$. Then the sequence $\{a_n\}$ is
   (A) not Cauchy (B) Cauchy but not convergent (C) convergent (D) not bounded.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and F be a primitive of f (i.e. $F' = f$). If $3x^2F(x) = f(x)$ for all $x \in \mathbb{R}$ then $f(x) =$
   (A) e^{x^3} (B) $3x^2e^{x^3}$ (C) $x^2e^{x^2}$ (D) $3xe^{x^3}$.

8. $1 \times 2 - 2 \times 3 + 3 \times 4 - 4 \times 5 + \cdots - (2022) \times (2023) =$
   (A) $(-2)(1011)(1012)$ (B) $-(1011)(1012)$
   (C) $(-4)(1011)(1012)$ (D) $2(1011)(1012)$.

9. The number of times the digit 7 is written while listing all integers from 1 to $1,00,000$ is
   (A) 10^4 (B) $5(10)^4 - 1$ (C) 10^5 (D) $5(10)^4$.

10. The differential equation $y'^2 - (x + \sin x)y' + x \sin x = 0$, with $y(0) = 0$ has
    (A) unique solution (B) two solutions (C) no solution (D) four solutions.

Part II

N.B. Each question in Part II carries 6 marks.

1. Consider $f(x) = x[x^2]$, where $[x^2]$ is the greatest integer less than or equal to x^2. Find the area of the region above X-axis and below $f(x), 1 \leq x \leq 10$.

2. In how many ways can numbers from 1 to 100 be arranged in a circle such that sum of pair of integers placed opposite each other is the same? (arrangements are equivalent up to rotation).

3. Find all triplets (x, y, z) of integers satisfying $x^2 + y^2 + z^2 = 16(x + y + z)$.

4. Suppose A is a singular matrix of order 3 with complex entries all of which having absolute value 1. Show that two rows or two columns of the matrix A are proportional.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f^3(x) = x$. Prove that $f^2(x) = x$.

Part III

1. Find [12]

   (a) $\lim_{n\to\infty} \dfrac{\gcd(1,6) + \gcd(2,6) + \cdots + \gcd(n,6)}{1 + 2 + \cdots + n}$;

   (b) $\lim_{n\to\infty} \dfrac{lcm(1,6) + lcm(2,6) + \cdots + lcm(n,6)}{1 + 2 + \cdots + n}$.

2. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 4$. [12]

   (a) Find the value of the determinant of a matrix $A = \begin{pmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{pmatrix}$.

   (b) Find the maximum and minimum value of the above determinant.

3. For every $t \in \mathbb{R}$, let L_t be the line segment joining $(0, 1)$ with $(t, 0)$. Suppose L_t intersects the parabola $y = x^2$ at the point P_t. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(t) = y$-coordinate of P_t. Answer the following questions with justification: [13]

   (a) Is f continuous?

   (b) Is f bounded?

   (c) What is $\lim_{t\to\infty} f(t)$?

   (d) Is f differentiable at 0?

4. The sequence $\{q_n(x)\}$ of polynomials is defined by $q_1(x) = 1 + x, q_2(x) = 1 + 2x$ and for $m \geq 1$ by

$$q_{2m+1}(x) = q_{2m}(x) + (m+1)xq_{2m-1}(x),$$
$$q_{2m+2}(x) = q_{2m+1}(x) + (m+1)xq_{2m}(x).$$

   Let x_n be the largest real solution of $q_n(x) = 0$. Prove that [13]

   (a) the sequence $\{x_n\}$ is increasing.

   (b) $x_{2m+2} > \dfrac{-1}{m+1}$ for $m \geq 1$.

   (c) the sequence $\{x_n\}$ converges to 0.

MADHAVA MATHEMATICS COMPETITION

Date: 29/01/2023 Max. Marks: 100

Solutions and Scheme of marking:

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The number of positive divisors of $2^{24} - 1$ is
   (A) 192 (B) 48 (C) 96 (D) 24
   Ans:(C)

2. The equation Re $(z^2) = 0$ represents
   (A) a circle (B) a pair of straight lines (C) an ellipse (D) a parabola
   Ans:(B)

3. If $A = \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$ and $\det A^3 = 125$, then the value of α is
   (A) ± 1 (B) ± 2 (C) ± 3 (D) ± 5.
   Ans: (C).

4. Let A, B, C be three non-collinear points in a plane. The number of points at a distance 1 from A, 2 from B and 3 from C is
   (A) exactly 1 (B) at most 1 (C) at most 2 (D) always 0.
   Ans: (B)

5. Let $A = \{x \in [-2, 3] : \cos x > 0\}$. Then
   (A) $\inf A = 0$ (B) $\sup A = \pi$ (C) $\inf A = -\pi/2$ (D) $\sup A = 3$.
   Ans: (C)

6. Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| \leq \dfrac{2023}{n}|a_n - a_{n-1}|, \ \forall n$. Then the sequence $\{a_n\}$ is
   (A) not Cauchy (B) Cauchy but not convergent (C) convergent (D) not bounded.
   Ans: (C)

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and F is a primitive of f (i.e. $F' = f$). If $3x^2 F(x) = f(x)$ for all $x \in \mathbb{R}$ then $f(x) =$
   (A) e^{x^3} (B) $3x^2 e^{x^3}$ (C) $x^2 e^{x^2}$ (D) $3xe^{x^3}$.
   Ans: (B)

8. $1 \times 2 - 2 \times 3 + 3 \times 4 - 4 \times 5 + \cdots - (2022) \times (2023) =$
   (A) $(-2)(1011)(1012)$ (B) $-(1011)(1012)$
   (C) $(-4)(1011)(1012)$ (D) $2(1011)(1012)$.
   Ans: (A)

9. The number of times the digit 7 is written while listing all integers from 1 to $1,00,000$ is
   (A) 10^4 (B) $5(10)^4 - 1$ (C) 10^5 (D) $5(10)^4$.
   Ans: (D)

10. The differential equation $y'^2 - (x + \sin x)y' + x \sin x = 0$, with $y(0) = 0$ has
    (A) unique solution (B) two solutions (C) no solution (D) four solutions.
    Ans: (B)

Part II

N.B. Each question in Part II carries 6 marks.

1. Consider $f(x) = x[x^2]$, where $[x^2]$ is the greatest integer less than or equal to x^2. Find the area of the region above X-axis and below $f(x), 1 \leq x \leq 10$.

   Solution:

   Observe that

   $$f(x) = \begin{cases} x, & 1 \leq x < \sqrt{2} \\ 2x, & \sqrt{2} \leq x < \sqrt{3} \\ : & : \\ 99x, & \sqrt{99} \leq x < 10 \\ 1000 & x = 10 \end{cases}$$ [4 Marks]

   Thus, the area of the required region is

   $$\sum_{k=1}^{99} \int_{\sqrt{k}}^{\sqrt{k+1}} kx dx = \sum_{k=1}^{99} \frac{k}{2} \left[x^2\right]_{\sqrt{k}}^{\sqrt{k+1}} = \sum_{k=1}^{99} \frac{k}{2} = 99 \times 25 = 2475$$ [2 Marks]

2. In how many ways can numbers from 1 to 100 be arranged in a circle such that sum of two integers placed opposite each other is the same? (arrangements are equivalent up to rotation.)

   Solution:

   It is clear that opposite pairs must be $(1, 100), (2, 99), \ldots, (50, 51)$ [1 Mark]

   1 can be placed anywhere and 100 must be placed opposite to 1. This can be done in exactly one way as all places are identical to start with.

   Now, 2 has 98 options and then 99 has to be placed opposite to 2.

   3 has 96 options and then 96 must be placed opposite to 3 and so on.

   By multiplication principle, the required number of ways is

   $98 \times 96 \times 94 \times \cdots \times 2 = 2^{49} \times 49!$. [5 Mark]

3. Find all triplets (x, y, z) of integers satisfying $x^2 + y^2 + z^2 = 16(x + y + z)$.

   Solution:

   Let (x, y, z) be a triplet satisfying the given condition.

   Thus, $x^2 + y^2 + z^2 = 16(x + y + z) \ldots\ldots (*)$

   Every square is congruent to $0, 1$ or 4 modulo 8, in fact, odd squares give remainder 1 when divided by 8. Since RHS of (*) is divisible by 8, so must be the LHS.

   Hence, x, y, z must be even integers. [3 Marks]

   Let $x = 2x_1$, $y = 2y_1$ and $z = 2z_1$, for some $x_1, y_1, z_1 \in \mathbb{Z}$.

   Substituting in (*), we get, $x_1^2 + y_1^2 + z_1^2 = 8(x_1 + y_1 + z_1) \ldots\ldots (**)$

   By similar argument as above, we must have x_1, y_1, z_1 must be even.

   Let $x_1 = 2x_2$, $y_1 = 2y_2$ and $z_1 = 2z_2$, for some $x_2, y_2, z_2 \in \mathbb{Z}$.

   Substituting in (**), we get, $x_2^2 + y_2^2 + z_2^2 = 4(x_2 + y_2 + z_2) \ldots\ldots (***)$

   Once again, arguing in similar manner, we must have x_2, y_2, z_2 to be even integers.

   Let $x_2 = 2x_3$, $y_2 = 2y_3$ and $z_2 = 2z_3$, for some $x_3, y_3, z_3 \in \mathbb{Z}$.

   Substituting in (***), we get, $x_3^2 + y_3^2 + z_3^2 - 2(x_3 + y_3 + z_3) = 0$.

   That is, $(x_3 - 1)^2 + (y_3 - 1)^2 + (z_3 - 1)^2 = 3$

   This implies, each of $|x_3 - 1|, |y_3 - 1|$ and $|z_3 - 1|$ must be 1.

   Hence, x_3, y_3, z_3 are either 2 or 0 and thus, x, y, z can be either 16 or 0.

   Thus, all possible triplets satisfying given condition are: $(16, 16, 16), (16, 16, 0), (16, 0, 16), (0, 16, 16), (0, 0, 16), (0, 16, 0), (16, 0, 0), (0, 0, 0)$. [3 Marks]

4. Suppose A is a singular matrix of order 3 with complex entries all of which having absolute value 1. Show that two rows or two columns of the matrix A are proportional.

Solution:
By suitable multiplication on each row and column, the matrix A can be written as $k\begin{pmatrix}1 & 1 & 1\\ 1 & a & b\\ 1 & c & d\end{pmatrix}$, where a, b, c, d, k are complex numbers of absolute value 1. [2 Marks]
The relation $\det(A) = 0$ gives $(a-1)(d-1) = (b-1)(c-1)$. Taking the complex conjugates of the above equation and using the fact that each number is of absolute value one, to get $\overline{ad}(a-1)(d-1) = \overline{bc}(b-1)(c-1)$. [2 Marks]
If $(a-1)(d-1) = 0$, then $(b-1)(c-1) = 0$ and two rows or columns are equal to $(1\ \ 1\ \ 1)$ and result is proved.
Suppose $(a-1)(d-1) = (b-1)(c-1) \neq 0$. Then $\overline{ad} = \overline{bc}$ and $ad = bc$. From $(a-1)(d-1) = (b-1)(c-1)$ we get $a+d = b+c$. Hence $\{a, d\} = \{b, c\}$ or $a = b = c = d$. It follows that the bottom two rows or the rightmost two columns are equal. [2 Marks]

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f^3(x) = x$. Prove that $f^2(x) = x$.
Solution:
We first prove that f is injective.
Since, $f(x) = f(y) \implies f^3(x) = f^3(y) \implies x = y$
hence, f is an injective function.
Also, f is continuous and hence it is monotonic. [2 Marks]
In case, f is increasing then for each $x \in \mathbb{R}$,
case(i) $x < f(x)$:
$x < f(x) \implies f(x) < f^2(x) \implies f^2(x) < f^3(x) = x$, which is a contradiction.
case (ii) $x > f(x)$:
$x > f(x) \implies f(x) > f^2(x) \implies f^2(x) > f^3(x) = x$, which is a contradiction.
Hence, if f is increasing then $f(x) = x$, for each $x \in \mathbb{R}$ and hence we get,
$f^2(x) = x$, $\forall x \in \mathbb{R}$. [2 Marks]
Now, let f be decreasing.
$x < y \implies f(x) > f(y) \implies f^2(x) < f^2(y)$ and thus f^2 is increasing.
case (i) $x < f^2(x)$:
$x < f^2(x) \implies f^2(x) < f^4(x) = f(x)$ i.e. $f^2(x) < f(x)$ and thus we get,
$f^4(x) < f^3(x) = x$. This gives, $x < f^2(x) < f^4(x) < f^3(x) = x$. This is a contradiction.
case (ii) $x > f^2(x)$:
$x > f^2(x) \implies f^2(x) > f^4(x)$ i.e. $f^2(x) > f(x)$ and thus we get,
$f^4(x) > f^3(x) = x$. This gives, $x > f^2(x) > f^4(x) > f^3(x) = x$. This is a contradiction.
Hence, we must have $f^2(x) = x$, for all $x \in \mathbb{R}$. [2 Marks]

Part III

1. Find [12]

(a) $\lim_{n\to\infty} \dfrac{\gcd(1,6) + \gcd(2,6) + \cdots + \gcd(n,6)}{1 + 2 + \cdots + n}$

(b) $\lim_{n\to\infty} \dfrac{lcm(1,6) + lcm(2,6) + \cdots + lcm(n,6)}{1 + 2 + \cdots + n}$

Solution:
(a) We first calculate the numerator.
$\sum_{n=6k+1}^{6k+6} \gcd(n,6) = \gcd(6k+1,6) + \gcd(6k+2,6) + \gcd(6k+3,6) + \gcd(6k+4,6) +$
$\gcd(6k+5,6) + \gcd(6k+6,6) = 1 + 2 + 3 + 4 + 5 + 6 = 15.$ [3 Marks]

Then $\sum_{k=1}^{m}\sum_{n=6k+1}^{6k+6}\gcd(n,6) = 15m$. Let $n = 6m$. Now

$$\lim_{n\to\infty}\frac{\gcd(1,6)+\gcd(2,6)+\cdots+\gcd(n,6)}{1+2+\cdots+n} = \lim_{m\to\infty}\frac{(15m)(2)}{6m(6m+1)} = 0.$$

[3 Marks]

(b) We first calculate the numerator.
$\sum_{n=6k+1}^{6k+6} lcm(n,6) = lcm(6k+1,6)+lcm(6k+2,6)+lcm(6k+3,6)+lcm(6k+4,6)+$
$lcm(6k+5,6)+lcm(6k+6,6) = 6(6k+1)+6(3k+1)+6(2k+1)+6(3k+2)+6(6k+5)+6(k+1) = 126k+66.$ [3 Marks]

Then $\sum_{k=1}^{m}(126k+66) = 66m+126\frac{m(m+1)}{2} = 66m+63m(m+1)$.
Let $n = 6m$. Now

$$\lim_{n\to\infty}\frac{lcm(1,6)+lcm(2,6)+\cdots+lcm(n,6)}{1+2+\cdots+n} = \lim_{m\to\infty}\frac{2[66m+63m(m+1)]}{6m(6m+1)} = \frac{63}{18} = \frac{7}{2}.$$

[3 Marks]

2. Let a, b, c be real numbers such that $a^2+b^2+c^2 = 4$. [12]

(a) Find the value of the determinant of a matrix $A = \begin{pmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{pmatrix}$.

(b) Find the maximum and minimum value of the above determinant.

Solution:
Let a, b, c be real numbers such that $a^2+b^2+c^2 = 4$. Consider

$$D = \begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix}$$

We add $R_1+(R_2+R_3)$

$$D = \begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix}$$

Now, $R_2-(c+a)R_1, R_3-(b+c)R_1$ gives

$$\begin{aligned} D &= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-c & b-a \\ 0 & a-b & a-c \end{vmatrix} \\ &= 2(a+b+c)\left((b-c)(a-c)+(a-b)^2\right) \\ &= 2(a+b+c)\left(a^2+b^2+c^2-ab-bc-ac\right) \end{aligned}$$

[5 Marks]

Let $s = a+b+c$. Hence, $s^2 = a^2+b^2+c^2+2(ab+bc+ac)$. [3 Marks]

Hence, $D = f(s) = 2s\left(4-\frac{s^2-4}{2}\right) = s(12-s^2) = 12s-s^3$.

Then $f'(s) = 12-3s^2 = 0$ gives $s = \pm 2$. Now $f''(s) = -6s$ is positive at $s = -2$ and negative at $s = 2$. Hence the maximum value of D is 16 at $(-2,0,0)$ and the minimum value of D is -16 at $(2,0,0)$. [4 Marks]

3. For each $t \in \mathbb{R}$ let L_t be the line segment joining the points $(0,1)$ and $(t,0)$. Let P_t be the point of intersection of the line segment L_t with the parabola $y = x^2$. Define function $f : \mathbb{R} \to \mathbb{R}$ as $f(t) = y$ coordinate of point P_t. Answer the following with justification. [13]

   (a) Is f a bounded function?

   (b) Is f a continuous function?

   (c) Find $\lim_{t\to\infty} f(t)$.

   (d) Is f differentiable at 0?

Solution :
The equation of line L_t is : $y = 1 - \frac{x}{t}$. Thus, $x-$ coordinate of the point of intersection of L_t with the parabola $y = x^2$ satisfy $1 - \frac{x}{t} = x^2$. This gives $x = -\dfrac{\frac{1}{t} \pm \sqrt{\frac{1}{t^2} + 4}}{2}$

For $t > 0$, we get $x = \dfrac{-1 + \sqrt{1 + 4t^2}}{2t} = \dfrac{2t}{1 + \sqrt{4t^2 + 1}}$

and thus, $y = x^2 = \dfrac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}}$.

Hence, $f(t) = \dfrac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}}$ for $t \geq 0$. [5 Marks]

Further, as $g(x) = x^2$ is an even function from $\mathbb{R}$ to $\mathbb{R}$, it can be observed by definition of f that f is an even function. Hence, for $t < 0$, we get,

$f(t) = f(-t) = \dfrac{4(-t)^2}{4(-t)^2 + 2 + 2\sqrt{4(-t)^2 + 1}} = \dfrac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}}$

Thus, we get, $f(t) = \dfrac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}}$, for all $t \in \mathbb{R}$. [3 Marks]

   (a) Since, $4t^2 < 4t^2 + 2 + 2\sqrt{4t^2 + 1}$, $\forall t \in \mathbb{R}$, we get, $0 \leq f(t) < 1$, $\forall t \in \mathbb{R}$.
   Hence, f is bounded. [1 Mark]

   (b) By continuity of polynomials, square-root function and algebra of continuous functions, function f is a continuous function. [1 Mark]

   (c) $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \dfrac{4t^2}{4t^2 + 2 + 2\sqrt{4t^2 + 1}} = \dfrac{4}{4} = 1.$ [1 Mark]

   (d) As $f(0) = 0$, Consider $\lim_{t\to 0} \dfrac{f(t)}{t} = \lim_{t\to 0} \dfrac{4t}{4t^2 + 2 + 2\sqrt{4t^2 + 1}} = 0$ and hence, f is differentiable at 0 and $f'(0) = 0$. [2 Marks]

4. The sequence $\{q_n(x)\}$ of polynomials is defined by $q_1(x) = 1 + x, q_2(x) = 1 + 2x$ and for $m \geq 1$ by

$$q_{2m+1}(x) = q_{2m}(x) + (m+1)xq_{2m-1}(x),$$
$$q_{2m+2}(x) = q_{2m+1}(x) + (m+1)xq_{2m}(x).$$

Let x_n be the largest real solution of $q_n(x) = 0$. Prove that [13]

   (a) the sequence $\{x_n\}$ is increasing.

   (b) $x_{2m+2} > \dfrac{-1}{m+1}$ for $m \geq 1$.

   (c) the sequence $\{x_n\}$ converges to 0.

Solution :
(a) Since each $q_n(x)$ has non-negative coefficients, any real zero cannot be positive. [1 Mark]

It is easy to check that $q_1(x)$ and $q_2(x)$ each have a negative zero. Suppose for some $n \geq 2$, it has been established that $q_i(0) > 0$ and that each $q_i(x)$ has at least one negative zero and that the greatest such zero x_i satisfies $x_{i-1} < x_i$ $(2 \leq i \leq n)$. Then for $1 \leq i \leq j \leq n$, $q_i(x) > 0$ for $x_j < x \leq 0$. From the recursion relations, it follows that $q_{n+1}(0) > 0$ and $q_{n+1}(x_n) < 0$, so that $q_{n+1}(x)$ has at least one zero in the interval $(x_n, 0)$. Thus, there is a largest real zero x_{n+1} and it satisfies $x_n < x_{n+1} < 0$. [3 Marks]
(b) From the recurrence relations, we find that

$$q_{2m+2}\frac{(-1)}{m+1} = -q_{2m-1}\frac{(-1)}{m+1}.$$

If $q_{2m+2}\dfrac{(-1)}{m+1} < 0$, then $x_{2m+2} > \dfrac{(-1)}{m+1}$. On the other hand, if $q_{2m+2}\dfrac{(-1)}{m+1} > 0$, then $q_{2m-1}\dfrac{(-1)}{m+1} < 0$, so that $x_{2m+2} > x_{2m-1} > \dfrac{(-1)}{m+1}$. [5 Marks]
(c) If r is any positive number, we can choose a positive integer m such that $-r < (-1)/(m+1) < 0$. Then for $n \geq 2m+2$, $-r < x_n < 0$. Hence the sequence $\{x_n\}$ converges to 0. [4 Marks]

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Bhaskaracharya Pratishthana, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date: 07/01/2024 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Find constants a, b such that $\lim_{x\to 0} \dfrac{\sqrt{ax+b}-2}{x} = 1$.
   (A) $a=-4, b=-4$ (B) $a=4, b=4$ (C) $a=-3, b=4$ (D) $a=4, b=-2$.

2. The value of n for which $i + 2(i)^2 + 3(i)^3 + 4(i)^4 + \cdots + n(i)^n$ equals $-16+15i$ is
   (A) 20 (B) 25 (C) 30 (D) 15.

3. Find the value of a real number b for which the sum of the squares of the zeros of $x^2-(b-2)x-b-1$ is minimal.
   (A) 2 (B) 3 (C) -5 (D) 1.

4. Let S be the set of all last two digits of the powers of 3. (For example, $03, 09, 27, 81, 43 \in S$) Then, the number of distinct elements of S is
   (A) 20 (B) 25 (C) 30 (D) 40.

5. The sum of the roots of $x^2 - 31x + 220 = 2^x(31 - 2x - 2^x)$ is
   (A) 10 (B) 7 (C) 3 (D) 4.

6. The number of regions in which the plane gets divided by curves $(\sin t, \sin 2t)$ and $(\cos t, \cos 2t)$ for $t \in \mathbb{R}$ is
   (A) 5 (B) 7 (C) 6 (D) 4.

7. In how many ways 6 persons can exchange seats among them in a row such that no one occupies his seat in a original position and exactly two of them have mutual exchange?
   (A) $\binom{6}{2} \times 6$ (B) $\binom{6}{2} \times 4!$ (C) $\binom{6}{2} \times 3$ (D) $\binom{6}{2} \times 4$.

8. Let $A = \{x \in (0,3) \ : \ [x]^2 = [x^2]\}$, where $[x]$ denotes the greatest integer less than or equal to x. Let $M = \sup A$. Then
   (A) $M \in A, M \notin (0,3)$ (B) $M \notin A, M \notin (0,3)$
   (C) $M \in A, M \in (0,3)$ (D) $M \in A, M \in (0,3)$.

9. If $4047 = x + \dfrac{x}{1+2} + \dfrac{x}{1+2+3} + \dfrac{x}{1+2+3+4} + \cdots + \dfrac{x}{1+2+3+\cdots+4047}$, then $x =$

   (A) 2000 (B) 2024 (C) 2002 (D) 2004.

10. Let $A(0,0), B(0,23), C(23,0)$ be the points in the plane. The number of points with integral coordinates that lie inside the triangle ABC (not on the boundary) is
    (A) 253 (B) 242 (C) 231 (D) 219.

Part II

N.B. Each question in Part II carries 6 marks.

1. Following operations are permitted with a quadratic $ax^2 + bx + c$

   (i) Switch a and c,

   (ii) Replace x by $x + t$ for $t \in \mathbb{R}$.

   Is it possible to convert $x^2 - x - 2$ into $x^2 - x - 1$ by using the operations permitted? Justify.

2. Consider a right angled triangle PRQ with coordinates of the vertices integers. If slope and length of the hypotenuse PQ are integers, then show that PQ is parallel to the X-axis.

3. Let $1 \leq a_1 < a_2 < \cdots < a_\ell \leq n$ be integers with $\ell > \dfrac{n+1}{2}$. Show that there exist i, j, k with $1 \leq i < j < k \leq \ell$ such that $a_i + a_j = a_k$.

4. The sequence (a_n) is defined by

$$a_1 = 0, |a_n| = |a_{n-1} + 1| \text{ for each } n \geq 2.$$

   For every positive integer n, prove that, $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq -\dfrac{1}{2}$.
   When does the equality hold?

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x(x-1)(2-x)$.

$$\text{Let } S = \{x \in \mathbb{R} : f(x+t) > f(x) \text{ for some } t > 0\}.$$

   (a) Draw the graph of f.

   (b) Is the set S non-empty? Justify.

   (c) Is the set S bounded? Find $\sup S$, if it exists.

Part III

1. Let $\mathbb{Z}_{10}$ denote the set of integers modulo 10.

   (a) i. Find a nonzero solution to the following system of equations in $\mathbb{Z}_{10}$: [2]

$$\begin{aligned} 4x + 6y &= 0 \\ 2x + 4y &= 0 \end{aligned}$$

   ii. Find a nonzero solution to the following system of equations in $\mathbb{Z}_{10}$: [2]

$$\begin{aligned} 4x + 3y &= 0 \\ x + 2y &= 0 \end{aligned}$$

   (b) Prove that the system of equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

   has a unique solution $x = 0, y = 0$ in $\mathbb{Z}_{10}$ if and only if the number $(ad - bc) \pmod{10} \in \{1, 3, 7, 9\}$. [8]

2. Let $f(x) = a_0 + a_1x + a_2x^2 + a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13}$ and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_{11}x^{11} + b_{12}x^{12} + b_{13}x^{13}$ be polynomials with real coefficients such that $a_{13} \neq 0, b_3 \neq 0$. Prove that the degree of $gcd(f, g) \leq 6$. [12]

3. (a) Let x_0 be an arbitrary real number. Define a sequence (x_n) as follows:

$$x_n = \frac{x_{n-1} + 4}{5}, \forall n \geq 1.$$

Show that sequence (x_n) is convergent. Find $\lim_{n\to\infty} x_n$. [5]

(b) Let n be a fixed positive integer. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero function satisfying following conditions:

i n^{th} derivative of f is continuous.

ii $f(5x + 3) = 5^n f\left(x + \frac{7}{5}\right), \forall x \in \mathbb{R}$.

Show that f is a polynomial of degree n. [8]

4. Let n be a positive integer greater than 1. Let $\rho(n)$ be the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

(a) Find $\rho(2)$ and $\rho(3)$. [4]

(b) Find $\rho(4)$. [3]

(c) Find $\rho(n)$ for each n. [6]

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Bhaskaracharya Pratishthana, Pune
and
Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai
Solutions and Scheme of Marking

Date: 07/01/2024 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Find constants a, b such that $\lim_{x\to 0} \dfrac{\sqrt{ax+b}-2}{x} = 1$.
(A) $a=-4, b=-4$ (B) $a=4, b=4$ (C) $a=-3, b=4$ (D) $a=4, b=-2$.
Ans: (B)

2. The value of n for which $i + 2(i)^2 + 3(i)^3 + 4(i)^4 + \cdots + n(i)^n$ equals $-16 + 15i$ is
(A) 20 (B) 25 (C) 30 (D) 15.
Ans: (C)

3. Find the value of a real number b for which the sum of the squares of the zeros of $x^2 - (b-2)x - b - 1$ is minimal.
(A) 2 (B) 3 (C) -5 (D) 1.
Ans: (D)

4. Let S be the set of all last two digits of the powers of 3. (For example, $03, 09, 27, 81, 43 \in S$) Then, the number of distinct elements of S is
(A) 20 (B) 25 (C) 30 (D) 40.
Ans: (A)

5. The sum of the roots of $x^2 - 31x + 220 = 2^x(31 - 2x - 2^x)$ is
(A) 10 (B) 7 (C) 3 (D) 4.
Ans: (B)

6. The number of regions in which the plane gets divided by curves $(\sin t, \sin 2t)$ and $(\cos t, \cos 2t)$ for $t \in \mathbb{R}$ is
(A) 5 (B) 7 (C) 6 (D) 4.
Ans: (C)

7. In how many ways 6 persons can exchange seats among them in a row such that no one occupies his seat in a original position and exactly two of them have mutual exchange?
(A) $\binom{6}{2} \times 6$ (B) $\binom{6}{2} \times 4!$ (C) $\binom{6}{2} \times 3$ (D) $\binom{6}{2} \times 4$.
Ans: (A)

8. Let $A = \{x \in (0,3) \mid [x]^2 = [x^2]\}$, where $[x]$ denotes the greatest integer less than or equal to x. Let $M = \sup A$. Then
(A) $M \in A, M \notin (0,3)$ (B) $M \notin A, M \notin (0,3)$
(C) $M \in A, M \in (0,3)$ (D) $M \notin A, M \in (0,3)$.
Ans: (D)

9. If $4047 = x + \dfrac{x}{1+2} + \dfrac{x}{1+2+3} + \dfrac{x}{1+2+3+4} + \cdots + \dfrac{x}{1+2+3+\cdots+4047}$, then $x =$

(A) 2000 (B) 2024 (C) 2002 (D) 2004.
Ans: (B)

10. Let $A(0,0), B(0,23), C(23,0)$ be the points in the plane. The number of points with integral coordinates that lie inside the triangle ABC (not on the boundary) is
(A) 253 (B) 242 (C) 231 (D) 219.
Ans: (C)

Part II

N.B. Each question in Part II carries 6 marks.

1. Following operations are permitted with a quadratic $ax^2 + bx + c$

   (i) Switch a and c

   (ii) Replace x by $x + t$ for $t \in \mathbb{R}$.

   Can you convert $x^2 - x - 2$ into $x^2 - x - 1$ with repeated operations (i) and (ii)?
   Solution: Let $f(x) = ax^2 + bx + c$ and let $g(x)$ be the polynomial obtained after performing both the above operations (i) and (ii) for some $t \in \mathbb{R}$. Then, $g(x) = cx^2 + (2tc + b)x + ct^2 + bt + a$.
   We observe that the discriminant of $f(x)$ and $g(x)$ is same. That is, the discriminant is invariant under the operations (i), (ii). [4]
   Now, the discriminant of $x^2 - x - 2$ is 9 whereas that of $x^2 - x - 1$ is 5. Thus, it is not possible to convert $x^2 - x - 2$ into $x^2 - x - 1$ with repeated operations (i) and (ii). [2]

2. Consider a right angled triangle PRQ with coordinates of the vertices integers. If slope and length of the hypotenuse PQ are integers, then show that PQ is parallel to the X-axis.
   Solution: Let $P(m_1, n_1)$ and $Q(m_2, n_2)$ be vertices with integer coordinates. The slope $\lambda = \dfrac{n_2 - n_1}{m_2 - m_1}$ is also integer. Suppose the length of the hypotenuse PQ is m which is also integer. Then $m^2 = (m_2 - m_1)^2 + (n_2 - n_1)^2 = (m_2 - m_1)^2 + \lambda^2(m_2 - m_1)^2 = (1 + \lambda^2)(m_2 - m_1)^2$. [4]
   Thus $(1 + \lambda^2)$ is a square implying that $\lambda = 0$. Therefore $n_1 = n_2$. Thus PQ is parallel to the X-axis. [2]

3. Let $1 \leq a_1 < a_2 < \cdots < a_\ell \leq n$ be integers with $\ell > \dfrac{n+1}{2}$. Show that there exist i, j, k with $1 \leq i < j < k \leq \ell$ such that $a_i + a_j = a_k$.

   The statement above is not true if n is an even integer.
   For example, if $n = 4$, choose $l = 3$ and consider the set $\{2, 3, 4\}$. Then the property claimed in the problem does not hold. In general, if $n = 2k$, choose $l = \lfloor \frac{2k+1}{2} \rfloor + 1$ and then the choice of elements $\{n - l + 1, n - l + 2 \cdots, n - 1, n\}$ gives a contradiction.
   The statement of the problem is correct if n is an odd integer:
   For an odd integer $n = 2k + 1$, let $\ell \geq \frac{n+1}{2} + 1$.
   Let $A = \{a_2, \cdots, a_\ell\}$ and $B = \{a_2 - a_1, a_3 - a_1, \cdots, a_\ell - a_1\}$.
   Then A and B are subsets of first n natural numbers with $|A| + |B| = 2\ell - 2 \geq n + 1$
   Therefore $A \cap B$ is non-empty. Hence there exist a_i, a_j such that $i \geq 2$ and $i \neq j$ such that $a_i = a_j - a_1$.

   Note :
   1. As there is an error in the problem, please give 6 marks to all the students.
   2. As per every year, top few papers will be centrally moderated. We shall re-assess the solutions of this problem at the time of moderation, and due credit to the correct answers/partial answers/counterexamples will be given at that time.

4. The sequence (a_n) is defined by

$$a_1 = 0, |a_n| = |a_{n-1} + 1| \text{ for each } n \geq 2.$$

For every positive integer n, prove that, $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq -\dfrac{1}{2}$.
When does the equality hold?

Solution 1: For each n, the number $\dfrac{a_1 + a_2 + \cdots + a_n}{n}$ is minimum if a_n is chosen to be a non-positive value of $(a_{n-1} + 1)$. [4]

Thus $\dfrac{a_1 + a_2 + \cdots + a_n}{n}$ is minimized if $a_1 = 0, a_2 = -1, a_3 = 0, a_4 = -1, \cdots$ [2]

Solution 2: Squaring each term, we get $a_1^2 = 0, a_2^2 = (a_1 + 1)^2, a_3^2 = (a_2 + 1)^2, \cdots$, $a_{n+1}^2 = (a_n + 1)^2$. Adding all these terms, we get
$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n+1}^2 = (a_1 + 1)^2 + (a_2 + 1)^2 + \cdots + (a_n + 1)^2$
$= a_1^2 + a_2^2 + \cdots + a_n^2 + 2(a_1 + a_2 + \cdots + a_n) + n.$
This implies that $a_{n+1}^2 = 2(a_1 + a_2 + \cdots + a_n) + n \geq 0$.

Hence $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq -\dfrac{1}{2}$. [4]

Equality holds if and only if n is even and the sequence is $\{0, -1, 0, -1. \cdots\}$ [2]

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x(x-1)(2-x)$.

$$\text{Let } S = \{x \in \mathbb{R} : f(x+t) > f(x) \text{ for some } t > 0\}.$$

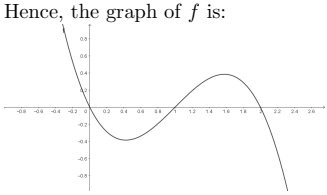
(a) Draw the graph of f.

(b) Is the set S non-empty? Justify.

(c) Is the set S bounded? Find $\sup S$, if it exists.

Solution:

(a) Observe: f is a polynomial of degree 3. Further, $f(x) > 0$ if $x < 0$ or $1 < x < 2$ and $f(x) < 0$ if $0 < x < 1$ or $x > 2$.
Hence, the graph of f is:



[1]

(b) $S \neq \emptyset$. Since, $f(1.1) = 1.1 \times 0.1 \times 0.9 > 0 = f(1)$, hence, $1 \in S$. [1]

(c) $f(x) = -x^3 + 3x^2 - 2x$. Thus, $f'(x) = -3x^2 + 6x - 2 = -3(x-1)^2 + 1$
For $x \geq 2$, $f'(x) \leq -2 < 0$ and hence, f is decreasing on $[2, \infty)$. Thus, $x \notin S$, $\forall x \geq 2$.
Hence, S is bounded above by 2.
$f'(x) = 0 \implies x = 1 \pm \dfrac{1}{\sqrt{3}}$. Let $x_1 = 1 - \dfrac{1}{\sqrt{3}}$ and $x_2 = 1 + \dfrac{1}{\sqrt{3}}$.
$f''(x) = -6x + 6$. $f''(x_1) > 0$ and $f''(x_2) < 0$, thus by Second derivative test, f has local minimum at x_1 and local maximum at x_2.
It can be seen that f is strictly decreasing on $(-\infty, x_1)$ and on (x_2, ∞) and f is strictly increasing on (x_1, x_2). Now, for any $x \leq 0$, we get, $f'(x) < -2$.
Let $n \in \mathbb{Z}$ be such that $n < n + 1 \leq 0$. Then, by Lagrange's mean value theorem, there exist $c \in (n, n+1)$ such that $f(n+1) - f(n) = f'(c) < -2$, as $c < 0$.

Hence, we get, $f(n) > f(n+1) + 2$, for all $n < n+1 \leq 0$.
Thus, as $n \to -\infty$, $f(n) \to \infty$.
$\implies$ there exist $n_0 \in \mathbb{Z}$ with $n_0 < 0$ such that $f(x) > f(x_2)$ for all $x \leq n_0$.
Hence, $\forall x < n_0$, we must have $x \notin S$. Thus, S is bounded below by n_0 and hence, S is a bounded set.
Claim: $\sup S = x_2$.
It is clear that x_2 is an upper bound of S.
Let $\varepsilon > 0$ be an arbitrary positive real number. Since, f has local maximum at x_2, we have $\delta > 0$ such that $f(x) \leq f(x_2)$, for all $x \in (x_2 - \delta, x_2 + \delta)$. Take $x = x_2 - \frac{\delta_1}{2}$ where $\delta_1 = \min\{\delta, \varepsilon, x_2 - x_1\}$. Then, as $x_1 < x < x_2$ and f is strictly increasing on (x_1, x_2), we get, $f(x) < f(x_2)$ and hence, we get, $x \in S$ such that $x_2 - \varepsilon < x \leq x_2$.
Thus, by characterization of supremum, $\sup S = x_2$. [4]

Part III

1. Let $\mathbb{Z}_{10}$ denote the set of integers modulo 10.

   (a) i. Find a nonzero solution to the following system of equations in $\mathbb{Z}_{10}$: [2]

   $$\begin{aligned} 4x + 6y &= 0 \\ 2x + 4y &= 0 \end{aligned}$$

   ii. Find a nonzero solution to the following system of equations in $\mathbb{Z}_{10}$: [2]

   $$\begin{aligned} 4x + 3y &= 0 \\ x + 2y &= 0 \end{aligned}$$

   (b) Prove that the system of equations

   $$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

   has a unique solution $x = 0, y = 0$ in $\mathbb{Z}_{10}$ if and only if the number $(ad - bc) \pmod{10} \in \{1, 3, 7, 9\}$. [8]

Solution:

(a) i. $x = 5, y = 0$ or any other correct solution [2]
ii. $x = 4, y = 8$ or any other correct solution [2]

(b) Consider the following system of equations where a, b, c, d are integers modulo 10.

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned} \tag{1}$$

Observe that $x = 0, y = 0$ is a solution of Eq. (1) for all possible values of a, b, c, d.

Claim: The system (1) has a unique solution $x = 0, y = 0$ if and only if $ad - bc$ is a unit modulo 10. (Note that the units modulo 10 are $1, 3, 7, 9$.)Suppose $ad - bc \in \{1, 3, 7, 9\}$ By multiplying the first equation by d and the second by b and then subtracting the one from the other, we get $(ad - bc)x = 0$. As $ad - bc$ is a unit, we then get $x = 0$. Similarly, we have $y = 0$. [2]

For the other implication, we have to make cases. Suppose $ad - bc$ is not a unit modulo 10.

We show that Eq. (1) has a solution $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e., there exists a nontrivial solution.

As $ad - bc$ is not a unit modulo 10, there exists $m \neq 0$ such that $m(ad - bc) = 0$ modulo 10. In fact m can be chosen to be 1, 2 or 5.

First observe that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} mb \\ -ma \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} md \\ -mc \end{pmatrix}$ are solutions of Eq. (1).

If at least one of the four terms $-ma, mb, -mc, md$ is nonzero modulo 10, we are done! We have to handle the case where $-ma = mb = -mc = md = 0$ modulo 10.

If $m = 1$, i.e. if $ad - bc = 0$ modulo 10, then in the above case, $a = b = c = d = 0$ and thus $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a nontrivial solution of Eq. (1) and we are done.

So we consider the case $ad - bc \neq 0$ modulo 10.

If $m = 2$. then in this case, each of $a, b, c, d \in \{0, 5\} \pmod{10}$. Thus $x = 2, y = 0$ is a nontrivial solution.
If $m = 5$, then in this case, each of $a, b, c, d \in \{0, 2, 4, 6, 8\} \pmod{10}$. Thus $x = 5, y = 0$ is a nontrivial solution. [6]

2. Let $f(x) = a_0 + a_1x + a_2x^2 + a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13}$ and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_{11}x^{11} + b_{12}x^{12} + b_{13}x^{13}$ be polynomials with real coefficients such that $a_{13} \neq 0, b_3 \neq 0$. Prove that the degree of $gcd(f, g) \leq 6$. [12]
Solution: Define $f_1(x) = a_0 + a_1x + a_2x^2, f_2(x) = a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3$ and $g_1(x) = b_0 + b_1x + b_2x^2 + b_3x^3, g_2(x) = b_{11}x + b_{12}x^2 + b_{13}x^3$. As $a_{13} \neq 0$ and $b_3 \neq 0$, we have degree$(f_2(x)) = 3$ and degree$(g_1(x)) = 3$, so that degree$(f_2(x)g_1(x)) = 6$. Now note that

$$f(x) = f_1(x) + x^{10}f_2(x),$$
$$g(x) = g_1(x) + x^{10}g_2(x).$$

[6]
Observe that $f(x)g_2(x) - g(x)f_2(x) = f_1(x)g_2(x) - g_1(x)f_2(x)$. $\cdots$ [I]
Note that the degree of the polynomial on RHS of [I] is equal to 6. If $h = \gcd(f, g)$, then h divides the polynomial on LHS of [I]. Therefore h divides the polynomial on RHS of [I]. Thus degree of $gcd(f, g) \leq 6$. [6]

3. (a) Let x_0 be an arbitrary real number. Define sequence (x_n) as follows:

$$x_n = \frac{x_{n-1} + 4}{5}, \forall n \geq 1.$$

Show that sequence (x_n) is convergent. Find $\lim_{n\to\infty} x_n$. [6]

(b) Let n be a fixed positive integer. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero function satisfying following conditions:

i n^{th} derivative of f is continuous.
ii $f(5x + 3) = 5^n f(x + \frac{7}{5}), \forall x \in \mathbb{R}$.

Show that f is a polynomial of degree n. [7]

Solution:

(a) Let $x_n = \frac{x_{n-1} + 4}{5}$. Observe that

$$x_{n+1} - x_n = \frac{x_n - x_{n-1}}{5} = \frac{x_{n-1} - x_{n-2}}{5^2} = \frac{x_1 - x_0}{5^n}.$$

This implies that the sequence (x_n) is Cauchy sequence of real numbers and hence convergent. Suppose the sequence (x_n) converges to c. Then by the given recurrence relation, we have $c = \frac{c+4}{5}$. Therefore $c = 1$. [6]

(b) Differentiating $f(5x+3) = 5^n f(x+\frac{7}{5})$ n times, we get that $5^n f^{(n)}(5x+3) = 5^n f^{(n)}(x+\frac{7}{5})$. Therefore $f^{(n)}(5x+3) = f^{(n)}(x+\frac{7}{5})$. Put $y = 5x+3$, then $x = (y-3)/5$. Therefore for all y,

$$f^{(n)}(y) = f^{(n)}\left(\frac{y-3}{5} + \frac{7}{5}\right) = f^{(n)}\left(\frac{y+4}{5}\right).$$

Put $y = x_0$. Now by using the sequence in part (a), we have $f^{(n)}(x_0) = f^{(n)}\left(\frac{x_0+4}{5}\right) = f^{(n)}(x_1)$. Continuing in this way, we get that $f^{(n)}(x_m)$ is a constant sequence. Since the sequence (x_m) converges to 1, by continuity of $f^{(n)}$ we have $f^{(n)}(x_m)$ converges to $f^{(n)}(1)$. Thus $f^{(n)}(x_m) = f^{(n)}(1)$, $\forall m$. Since x_0 is an arbitrary real number, $f^{(n)}(x)$ is a constant function. Hence f is a polynomial of degree at most n. However by comparing the coefficient of highest degree term in the equation $f(5x+3) = 5^n f(x+\frac{7}{5})$, we get that f must be a polynomial of degree n.

[7]

4. Let n be a positive integer bigger than 1. Let $\rho(n)$ be the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

(a) Find $\rho(2)$ and $\rho(3)$. [4]

(b) Find $\rho(4)$. [5]

(c) Find $\rho(n)$ for each n. [4]

Solution:

(a) For $n = 2$, the determinant of such a matrix is negative. Therefore $\rho(2) = 2$. [1]
Let $n = 3$.
Claim: All three rows R_1, R_2, R_3 are linearly independent.
Suppose $c_1R_1 + c_2R_2 + c_3R_3 = 0$. In the first column, first entry is zero and all other entries are positive. Therefore, we get that either c_2, c_3 have opposite signs or both are zero. The same argument applies to the pairs c_1, c_2 and c_1, c_3. Hence they all must be zero. Hence $\rho(3) = 3$. [3]

(b) Let $n = 4$.
By similar argument as in the case of $n = 3$, the first three rows are linearly independent.
Consider a 4×4 matrix $A = ((i-j)^2)_{i,j=1}^4$. Note that rank of A is 3. Hence $\rho(4) = 3$. (Any other correct example). [5]

(c) We show that for all $n \geq 3, \rho(n) = 3$.
By similar argument as in the case of $n = 3$, the first three rows are linearly independent.
We now present an example of a matrix of rank at most 3.
Consider an $n \times n$ matirx $A = ((i-j)^2)_{i,j=1}^n$.
Note that $A = \begin{pmatrix} 1^2 \\ 2^2 \\ \vdots \\ n^2 \end{pmatrix} (1, 1, \cdots, 1) - 2 \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} (1, 2, \cdots, n) + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (1^2, 2^2, \cdots, n^2)$.
Since A is the sum of three matrices of rank 1, the rank of A is at most 3. [4]
(Any other correct example).

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Bhaskaracharya Pratishthana, Pune

Date: 12/01/2025 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The values of a and b for which $(x-1)^2$ divides ax^4+bx^3+1 are
   (A) $a=-2, b=4$ (B) $a=3, b=-4$ (C) $a=-3, b=4$ (D) $a=2, b=-3$.

2. If $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(x)-f(y)|\le|x-y|^2$ for all $x,y\in\mathbb{R}$ and $f(1)=5$ then $f(2025)$ is
   (A) 1 (B) 5 (C) 2025 (D) 0.

3. Let $z\in\mathbb{C}$. The area of triangle whose vertices are represented by $-z, iz, z-iz$ is
   (A) $(1/2)|z|$ (B) $|z|$ (C) $(3/2)|z|^2$ (D) $|z|^2$.

4. The remainder when $x^{100}-2x^{51}+1$ is divided by x^2-1 is
   (A) $x-2$ (B) $2x-1$ (C) $-2x+2$ (D) $x-1$.

5. If $f(x)+2f(1-x)=x^2+2$ for all real numbers x and f is differentiable function, then the value of $f'(8)$ is
   (A) 0 (B) -4 (C) -3 (D) 4.

6. If the function f is periodic and for some fixed $a>0$ and for all real numbers x we have
   $f(x+a)=\dfrac{1+f(x)}{1-f(x)}$, then the possible value of period is
   (A) $4a$ (B) a (C) $2a$ (D) $8a$.

7. For real numbers a, b if one root of the equation $(a-b)x^2+ax+1=0$ is double the other, then the greatest value of b is
   (A) 9/8 (B) 8/9 (C) 8 (D) 9.

8. The value of $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^4+n^2+1}$ is
   (A) 1/3 (B) 1/4 (C) 1/5 (D) 1/2.

9. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(r+\frac{1}{n})=f(r)$ for all rational numbers r and positive integers n. Which of the following is true?
   (A) Image of f is uncountable (B) Image of f is a singleton set
   (C) Image of f is two point set (D) such a function does not exist.

10. The number of regions the curves $y=x^3$ and $y=\dfrac{x^2}{x+1}$ divide the square $[0,1]\times[0,1]$ is
    (A) 2 (B) 3 (C) 4 (D) 5

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(f(x)) = (f(x))^2$ for all $x \in \mathbb{R}$ and $f(100) = 200$. Find all possible values of $f(500)$.

2. Let $A = \left\{ \dfrac{2x+5}{3x-1} : x < 0 \right\}, B = \left\{ x : \dfrac{2x+5}{3x-1} < 0 \right\}$. Find $\sup A, \inf A, \sup B, \inf B$ if they exist. Justify your answer.

3. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with integer coefficients such that $p(0) \neq 0$ and $p(r) = p(s) = 0$ for two integers $0 < r < s$. Prove that for some k, $a_k \leq -s$.

4. For any positive integer m, show that there exists a real $m \times m$ matrix A such that $A^3 = A + I$. Also, show that for any such A, determinant of A is positive.

5. A frog starts at the point $(0, 0)$ in the coordinate plane and makes a sequence of jumps. In every jump frog covers a distance of 10 units and after each jump the frog is at a point whose coordinates are both integers.

   (a) Show that the frog can never reach the point $(2025, 2025)$.

   (b) Show that the frog can reach the point $(2026, 2026)$. Find the minimum number of jumps needed for the frog to achieve this.

Part III

1. Let $\{s_n\}$ be a sequence of real numbers and let $\{t_n\}$ be a sequence defined by $t_k = s_{k+1} - s_k$ and $t_{k+1} - t_k = 1$ for all $k \in \mathbb{N}$.

   (a) Find s_1 if $s_8 = s_{10} = 0$. [2]

   (b) Find s_1 if $s_{20} = s_{25} = 0$. [4]

   (c) Let $s_n = s_m = 0$ for some distinct positive integers m, n. Prove that $s_k \in \mathbb{Z}$ for all $k \in \mathbb{N}$ if and only if m, n are of different parity. [6]

2. For $0 < k \leq 1, n \in \mathbb{N}$, let $p_n(x) = x^n + x^{n-1} + \cdots + x - k$.

   (a) Show that for each n, $p_n(x)$ has a unique positive real root. [2]

   (b) If a_n is the positive root of $p_n(x)$, then show that the sequence $\{a_n\}$ is convergent. [5]

   (c) Find $\lim\limits_{n\to\infty} a_n$. [5]

3. Let $L := \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$.

   (a) Show that there is no regular hexagon with all its vertices in L. [5]

   (b) Show that for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that an equilateral triangle with two of its vertices at $(0, 0)$ and $(2n, 0)$ respectively has its third vertex inside an ε-neighbourhood of a point in L. [7]

4. Let A be a square matrix of order $2k$ with entries 1 to $4k^2$ in some order exactly once.

   (a) Show that there exists a row or a column having two entries with their difference at least $2k^2$. [3]

   (b) Show that there exists a row or a column having two entries with their difference at least $2k^2 + k - 1$. [7]

   (c) For $k = 2$, find such an A with the difference between any two entries in same row or column is at most 9. [4]

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Date: 12/01/2025 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.

Solutions and Scheme of Marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. The values of a and b for which $(x-1)^2$ divides ax^4+bx^3+1 are
(A) $a=-2, b=4$ (B) $a=3, b=-4$ (C) $a=-3, b=4$ (D) $a=2, b=-3$.
Ans: (B)

2. If $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(x)-f(y)|\le|x-y|^2$ for all $x,y\in\mathbb{R}$ and $f(1)=5$ then $f(2025)$ is
(A) 1 (B) 5 (C) 2025 (D) 0.
Ans: (B)

3. Let $z\in\mathbb{C}$. The area of triangle whose vertices are represented by $-z, iz, z-iz$ is
(A) $(1/2)|z|$ (B) $|z|$ (C) $(3/2)|z|^2$ (D) $|z|^2$.
Ans: (C)

4. The remainder when $x^{100}-2x^{51}+1$ is divided by x^2-1 is
(A) $x-2$ (B) $2x-1$ (C) $-2x+2$ (D) $x-1$.
Ans: (C)

5. If $f(x)+2f(1-x)=x^2+2$ for all real numbers x and f is differentiable function, then the value of $f'(8)$ is
(A) 0 (B) -4 (C) -3 (D) 4.
Ans: (D)

6. If the function f is periodic and for some fixed $a>0$ and for all real numbers x we have
$f(x+a)=\dfrac{1+f(x)}{1-f(x)}$, then the possible value of period is
(A) $4a$ (B) a (C) $2a$ (D) $8a$.
Ans: (A)

7. For real numbers a, b if one root of the equation $(a-b)x^2+ax+1=0$ is double the other, then the greatest value of b is
(A) 9/8 (B) 8/9 (C) 8 (D) 9.
Ans: (A)

8. The value of $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^4+n^2+1}$ is
(A) 1/3 (B) 1/4 (C) 1/5 (D) 1/2.
Ans: (D)

9. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(r+\frac{1}{n})=f(r)$ for all rational numbers r and positive integers n. Which of the following is true?
(A) Image of f is uncountable (B) Image of f is a singleton set
(C) Image of f is two point set (D) such a function does not exist.
Ans: (B)

10. The number of regions the curves $y = x^3$ and $y = \dfrac{x^2}{x+1}$ divide the square $[0,1] \times [0,1]$ is
(A) 2 (B) 3 (C) 4 (D) 5
Ans: (C)

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(f(x)) = (f(x))^2$ for all $x \in \mathbb{R}$ and $f(100) = 200$. Find all possible values of $f(500)$.
Solution: Let $f(f(x)) = (f(x))^2$. Put $x = 100$. Then we have
$f(f(100)) = f(200) = (f(100))^2 = (200)^2$. Now by the Intermediate Value Property, $[200, (200)^2] \subseteq R(f)$. [4]
As $500 \in [200, (200)^2]$, there exists a real number a such that $f(a) = 500$. Note that $f(f(a)) = f(500) = (f(a))^2 = (500)^2$. [2]

2. Let $A = \left\{ \dfrac{2x+5}{3x-1} : x < 0 \right\}, B = \left\{ x : \dfrac{2x+5}{3x-1} < 0 \right\}$. Find $\sup A, \inf A, \sup B, \inf B$ if they exist. Justify your answer.
Solution: $A = \left\{ \dfrac{2x+5}{3x-1} : x < 0 \right\}$
Let $f : (-\infty, 0) \to \mathbb{R}$ be a function defined as $f(x) = \dfrac{2x+5}{3x-1}$.
Then $f'(x) = \dfrac{-17}{(3x-1)^2} < 0$ and hence f is a decreasing function on $(-\infty 0)$.
Thus, $\sup(A) = \sup(f) = \lim\limits_{x \to 0} f(x) = -5$
Also, $\inf(A) = \inf(f) = \lim\limits_{x \to -\infty} f(x) = \dfrac{2}{3}$ [3]
$B = \left\{ x \in \mathbb{R} : \dfrac{2x+5}{3x-1} < 0 \right\}$.
Note that $x \in B \iff (2x+5 > 0$ and $3x-1 < 0)$ OR $(2x+5 < 0$ and $3x-1 > 0)$
Hence, $x \in B \iff (x > -5/2$ and $x < 1/3)$ OR $(x < -5/2$ and $x > 1/3)$
Thus, $x \in B \iff x > -5/2$ and $x < 1/3$.
So, $B = \left(-\dfrac{5}{2}, \dfrac{1}{3} \right)$ and hence, $\inf(B) = -\dfrac{5}{2}$ and $\sup(B) = \dfrac{1}{3}$. [3]

3. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with integer coefficients such that $p(0) \neq 0$ and $p(r) = p(s) = 0$ for two integers $0 < r < s$. Prove that for some k, $a_k \leq -s$.
Solution: Since $p(s) = 0$, we have $p(x) = (x - s)q(x)$, where
$q(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}, b_i$ are integers. [2]
Observe that $a_0 = -sb_0$. Thus $b_0 \neq 0$. If $b_0 > 0$, then $a_0 < -s$ and we are through. Now let $b_0 < 0$. As $q(x)$ has at least one real positive root, by Descarte's rule of sign, there is at least one sign change in the coefficients of $q(x)$. Thus there exists k such that $b_k \leq 0$ and $b_{k+1} > 0$. This implies $a_{k+1} = -sb_{k+1} + b_k \leq -s$. [4]

4. For any positive integer m, show that there exists a real $m \times m$ matrix A such that $A^3 = A + I$. Also, show that for any such A, determinant of A is positive.
Solution: Note that the matrix $A = \lambda I$ satisfies $A^3 = A + I$ if and only if $\lambda^3 = \lambda + 1$. Since $\lambda^3 - \lambda - 1$ is a cubic polynomial, it has at least one real root. Therefore such matrix A exists. [2]
Claim: The minimal polynomial of A has exactly one real root and two complex roots. Let $p(x) = x^3 - x - 1$. Since $p(1) < 0$ and $p(2) > 0$, by Intermediate Value Property, there is a root between 1 and 2. Note that the sum of the roots is zero. Therefore all roots cannot be positive. If $-1 < x < 0$, then $x^3 < 0$ but $x + 1 > 0$. Therefore $p(x)$ has no root in $(-1, 0)$. If $x < -1$, then $p'(x) = 3x^2 - 1 > 0$, and $p(-1) = -1$. Hence

$p(x) < -1$ for all $x < -1$. Therefore $p(x)$ has no root in $(-\infty, -1)$. Hence $p(x)$ has exactly one real positive root say α and two complex roots $\beta, \overline{\beta}$. If the multiplicity of α is r and of β is s, then $\det A = (\alpha)^r(\beta\overline{\beta})^s > 0$. [4]

5. A frog starts at the point $(0, 0)$ in the coordinate plane and makes a sequence of jumps. In every jump frog covers a distance of 10 units and after each jump the frog is at a point whose coordinates are both integers.

   (a) Show that the frog can never reach the point $(2025, 2025)$.

   (b) Show that the frog can reach the point $(2026, 2026)$. Find the minimum number of jumps needed for the frog to achieve this.

   Solution: Note that by the given condition, the only points in $\mathbb{R}^2$ where frog can reach in one jump from $(0, 0)$ are

   $$(\pm 10, 0), (0, \pm 10), (\pm 6, \pm 8), (\pm 8, \pm 6).$$

   Thus, each jump corresponds to adding one of the above vectors to the previous position of the frog.

   (a) Observe that after each jump, both the coordinates of frog's position remains even. Hence, frog cannot reach to $(2025, 2025)$. [2]

   (b) Observe that $(2026, 2026) = (0, 0) + 144(6, 8) + 144(8, 6) + (10, 0) + (0, 10)$
   Thus, the frog will reach to $(2026, 2026)$ by taking 290 jumps from $(0, 0)$.
   Let $s = x + y$ be the sum of x and y coordinates of the frog's position (x, y) in $\mathbb{R}^2$. Thus, it starts at $s = 0$ and ends at $s = 4052$. Further, by observing the possible jumps, each jump changes the sum of x and y coordinates by atmost 14 and $4052 > 4046 = 289 \times 14$. Thus, it follows that the frog must take more than 289 jumps to reach to the destination. Hence, the minimum number of jumps needed for the frog to reach $(2026, 2026)$ from $(0, 0)$ is 290. [4]

Part III

1. Let $\{s_n\}$ be a sequence of real numbers and let $\{t_n\}$ be a sequence defined by $t_k = s_{k+1} - s_k$ and $t_{k+1} - t_k = 1$ for all $k \in \mathbb{N}$.

   (a) Find s_1 if $s_8 = s_{10} = 0$. [2]

   (b) Find s_1 if $s_{20} = s_{25} = 0$. [4]

   (c) Let $s_n = s_m = 0$ for some distinct positive integers m, n. Prove that $s_k \in \mathbb{Z}$ for all $k \in \mathbb{N}$ if and only if m, n are of different parity. [6]

   Solution: Let $\{s_n\}$ be a sequence of real numbers and let $\{t_n\}$ be a sequence defined by $t_k = s_{k+1} - s_k$ and $t_{k+1} - t_k = 1$ for all $k \in \mathbb{N}$. Substituting t_k, we get $s_{k+2} - 2s_{k+1} + s_k = 1$. [I]

   (a) Putting $k = 8$ in [I], we have $s_{10} - 2s_9 + s_8 = 1$. Given that $s_8 = s_{10} = 0$. Thus we get $s_9 = -1/2$. Now putting $k = 7$ in [I], we have $s_7 = 3/2$. Now by successively putting values $k = 6, 5, 4, 3, 2, 1$; we get $s_1 = 63/2$. [2]

   (b) Given that $s_{20} = s_{25} = 0$. We have $s_{21} = s_{20} + t_{20}$,
   $s_{22} = s_{21} + t_{21} = s_{21} + t_{20} + 1 = s_{20} + 2t_{20} + 1$,
   $s_{23} = s_{22} + t_{22} = s_{20} + 2t_{20} + 1 + t_{22} = s_{20} + 2t_{20} + 1 + t_{20} + 2 = s_{20} + 3t_{20} + 3$.
   Similarly, $s_{24} = s_{20} + 4t_{20} + 6$ and $s_{25} = s_{20} + 5t_{20} + 10$. Now putting $s_{20} = s_{25} = 0$, we have $t_{20} = -2$. Thus $t_{19} = -3$ and $s_{19} = s_{20} - t_{20} = 2$. Now $s_{18} = s_{19} - t_{19} = 2 + 3 = 5$. One may now use [I] and successively find values of s_k for $k < 18$. We then get $s_1 = 228$. [4]

(c) One can use the sets
$\{t_k\} = \{t_1, t_1+1, t_1+2, \cdots, t_1+(k-1), \cdots\}$
$\{s_k\} = \{s_1, s_1+t_1, s_1+2t_1+1, s_1+3t_1+1+2, \cdots\}$
$s_k = s_1 + (k-1)t_1 + (1+2+\cdots+(k-2)) = s_1 + (k-1)t_1 + \frac{(k-2)(k-1)}{2}$
Thus $s_n = s_1 + (n-1)t_1 + \frac{(n-2)(n-1)}{2}$ and $s_m = s_1 + (m-1)t_1 + \frac{(m-2)(m-1)}{2}$.
Taking difference, we get
$0 = s_n - s_m = t_1(n-m) + \frac{(n-2)(n-1)-(m-2)(m-1)}{2} = t_1(n-m) + \frac{(n-m)(n+m-3)}{2}$
Therefore, $t_1 = \frac{3-n-m}{2}$. Hence, t_1 is an integer if and only if m, n are of different parity. Now s_k is an integer if and only if t_1 is integer. [6]

2. For $0 < k \leq 1, n \in \mathbb{N}$, let $p_n(x) = x^n + x^{n-1} + \cdots + x - k$.

(a) Show that for each n, $p_n(x)$ has a unique positive real root. [2]

(b) If a_n is the positive root of $p_n(x)$, then show that the sequence $\{a_n\}$ is convergent. [5]

(c) Find $\lim_{n\to\infty} a_n$. [5]

Solution: For $0 < k \leq 1$ and for each $n \in \mathbb{N}$, we have $p_n(x) = x^n + x^{n-1} + \cdots + x - k$.

(a) Note that $p_n(0) = -k < 0$ and $p_n(k) > 0$. Thus, by intermediate value theorem, $p_n(x)$ must have a root in $(0, k)$. Further, $p_n'(x) > 0$ for all $x > 0$. Thus, $p_n(x)$ is increasing in $(0, \infty)$ and hence $p_n(x)$ has unique positive root for each n. [2]

(b) Note that $p_{n+1}(x) = p_n(x) + x^{n+1}$. Thus $P_{n+1}(a_n) = a_n^{n+1} > 0$. Thus $a_{n+1} \in (0, a_n)$ for each n. Thus, sequence (a_n) is monotonically decreasing. [5]
Further, we know that $a_n > 0\ \forall n \in \mathbb{N}$. Thus, (a_n) is a convergent sequence.

(c) Note that $p_n(x) = \dfrac{x(1-x^n)}{1-x} - k = \dfrac{x^{n+1} - (k+1)x + k}{x-1}$
Thus, $p_n(a_n) = 0$ gives $\dfrac{a_n^{n+1}}{k+1} = a_n - \dfrac{k}{k+1}$.
Further, we have, $0 < a_n < a_2 \leq 1,\ \forall n \geq 2$. Hence, $0 < a_n^{n+1} \leq a_2^{n+1},\ \forall n \geq 2$.
Hence the sequence (a_n^{n+1}) converges to 0 implying $\lim_{n\to\infty} a_n = \dfrac{k}{k+1}$. [5]

3. Let $L := \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$.

(a) Show that there is no regular hexagon with all its vertices in L. [5]

(b) Show that for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that an equilateral triangle with two of its vertices at $(0, 0)$ and $(2n, 0)$ respectively has its third vertex inside an ε-neighbourhood of a point in L. [7]

Solution: Let $L := \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$.

(a) If possible, let $ABCDEF$ be a regular hexagon with all its vertices in L. Since all of its vertices have integer coordinates, by Pick's Theorem we get that the area of the hexagon is given by

$$[ABCDEF] = i + \frac{b}{2} - 1$$

where i and b are the numbers of points in L in the interior and on the boundary of $ABCDEF$ respectively. This gives us that the area of hexagon $ABCDEF$ is a rational number. On the other hand, we know that for a regular hexagon, the area is also given by

$$[ABCDEF] = \frac{3\sqrt{3}}{2} AB^2$$

Since A, B are points with integer coordinates, AB^2 is also an integer. This gives us that the area of $ABCDEF$ is an irrational number, which contradicts our earlier conclusion. Hence there can be no regular hexagon with all its vertices in L. [5]

(b) Note that if an equilateral triangle has two of its vertices at $(0,0)$ and $(2n,0)$ respectively for some $n \in \mathbb{N}$, then its third vertex must be at $(n, n\sqrt{3})$. Hence the problem reduces to showing that for any $\epsilon > 0$, there exist $n, k \in \mathbb{N}$ such that $|n\sqrt{3} - k| < \epsilon$, or equivalently that, there exists $n \in \mathbb{N}$ such that $\{n\sqrt{3}\} < \epsilon$, where $\{\cdot\}$ represents the fractional part function. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \epsilon$. We divide the unit interval $I := [0,1)$ into N equal parts $I_1, I_2, \ldots, I_N$ of size $\frac{1}{N}$ each. Hence, we have that $I_k := [(k-1)/N, k/N)$ for each $k \in \{1, 2, \ldots, N\}$. Consider the set $M := \{n\sqrt{3} \mid n \in \mathbb{N}\}$. Since M is an infinite set, by Pigeon-Hole Principle, there exist $n_1, n_2 \in \mathbb{N}$ such that $\{n_1\sqrt{3}\}, \{n_2\sqrt{3}\} \in I_{k_0}$ for some $k_0 \in \{1, 2, \ldots, N\}$. Without loss of generality let $n_1 > n_2$. Then in this case we see that $\{(n_1 - n_2)\sqrt{3}\} \in I_1$. Since the length of I_1 is $1/N$ which is smaller than ϵ, this completes the proof. [7]

4. Let A be a square matrix of order $2k$ with entries 1 to $4k^2$ in some order exactly once.

(a) Show that there exists a row or column having two entries with their difference at least $2k^2$. [3]

(b) Show that there exists a row or column having two entries with their difference at least $2k^2 + k - 1$. [7]

(c) For $k = 2$, find such an A with the difference between any two entries in same row or column is at most 9. [4]

Solution:

(a) If 1 and $4k^2$ are in the same row or column, then $4k^2 - 1 \geq 2k^2$. Otherwise suppose a is in the intersection of row containing 1 and column containing $4k^2$. Then $\max\{4k^2 - a, a - 1\} > \frac{4k^2-1}{2}$. Therefore $\max\{4k^2 - a, a - 1\} \geq 2k^2$. [3]

(b) Define sets S and T as follows:
$S = \{(i,j) : a_{ij} \in \{1, 2, \cdots, k^2\}\}$ and $T = \{(i,j) : \textit{either } a_{ip} \textit{ or } a_{qj} \in \{1, 2, \cdots, k^2\}\}$.
That is T is a union of rows and columns in which there is an entry from $\{1, 2, \cdots, k^2\}$.
Case 1: Suppose that the entries in S forms a $k \times k$ submatrix of A.
Let $M = \max\{a_{ij} : (i,j) \in T\}$. As $(i,j) \in T$, either i^{th} row or j^{th} column has an entry from $\{1, 2, \cdots, k^2\}$. But, the entries in S is a $k \times k$ submatrix. So either i^{th} row or j^{th} column has k entries from $\{1, 2, \cdots, k^2\}$. Thus some difference is at least $M - (k^2 - k + 1)$ (because largest k entries are $k^2, k^2 - 1, k^2 - 2, \cdots, k^2 - k + 1$). In this case, we now show that $M \geq 3k^2$. This will follow if we show that $|T| \geq 3k^2$. Suppose T is a union of a rows and b columns, then $|T| = 2ka + 2kb - ab$. But $|S| = k^2 \leq ab$. Therefore $|T| = 2ka + 2kb - ab = 2ka + b(2k - a) \geq 2ka + \frac{k^2}{a}(2k - a)$ $= 2k(a + \frac{k^2}{a}) - k^2 \geq 2k(2k) - k^2 = 3k^2$.
Thus some difference is at least $3k^2 - (k^2 - k + 1) = 2k^2 + k - 1$.
Case 2: Suppose that the entries in S do not form a $k \times k$ submatrix of A. In this case, we show that $|T| \geq 3k^2 + k$. Suppose $a \geq b$. As the entries in S do not not a $k \times k$ submatrix, $a \geq k + 1$. Suppose $a = k + r \geq k + 1$. As $ab \geq k^2$, we have $b \geq k - r + 1$. But $|T| = 2ka + b(2k - a) \geq 2k(k + r) + (k - r + 1)(2k - k - r) =$ $3k^2 + k + r(r - 1) \geq 3k^2 + k$.
Thus some difference is at least $3k^2 + k - k^2 = 2k^2 + k$. [7]

(c) One possible matrix is $\begin{pmatrix} 10 & 12 & 14 & 16 \\ 9 & 11 & 13 & 15 \\ 2 & 4 & 6 & 8 \\ 1 & 3 & 5 & 7 \end{pmatrix}$ [4]

MADHAVA MATHEMATICS COMPETITION
(A Mathematics Competition for Undergraduate Students)
Organized by
Bhaskaracharya Pratishthana, Pune

Date: 11/01/2026 Max. Marks: 100
Time: 12.00 noon to 3.00 p.m.
N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let $a < b < c$ be three positive integers in geometric progression whose common ratio is also an integer. If the arithmetic mean of a, b, c is $b + 2$, then the value of a is
(A) 24 (B) 12 (C) 6 (D) 4.

2. Triangles are formed by joining the vertices of a regular polygon of 12 sides. The number of triangles having no side in common with the polygon is
(A) 124 (B) 220 (C) 96 (D) 112.

3. If $f(x) = \begin{cases} ax + b & \text{if } x < -1 \\ x^2 + 1 & \text{if } -1 \le x < 1 \\ 3x - 1 & \text{if } x \ge 1, \end{cases}$
is continuous everywhere, then which of the following values can ab never take?
(A) -3 (B) -1 (C) 0 (D) 3.

4. Let $f : [1, 3] \to \mathbb{R}$ be a continuous function. If f takes rational values for all x and $f(2) = 10$, then $f(1.5) =$
(A) 7.5 (B) 10 (C) 1.5 (D) 5.

5. If $f(x) = (x + 1)^2 - 1,\ x \ge -1$ then the set $\{x : f(x) = f^{-1}(x)\}$ is
(A) ϕ (B) $\{-1, 1\}$ (C) $\{0, 1\}$ (D) $\{-1, 0\}$.

6. The number of real solutions of $1 + |e^x - 1| = e^x(e^x - 2)$ is
(A) 1 (B) 2 (C) 0 (D) 3.

7. In how many ways can you choose an odd number of objects from n objects?
(A) 2^{n-1} (B) 2^n (C) $2^n - 1$ (D) n.

8. The family of circles with center $(a, 0)$ and radius a satisfies the differential equation
(A) $\frac{dy}{dx} + x^2 - y^2 = 0$ (B) $2xy\frac{dy}{dx} + x^2 - y^2 = 0$ (C) $2xy\frac{dy}{dx} + x - y = 0$ (D) $2xy\frac{dy}{dx} + x^2 + y^2 = 0$.

9. The series $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(x^{2n}) + \sin(x^{2n+1}) - e^{-|x|n}}{1 + n^2}$
(A) converges pointwise everywhere on $\mathbb{R}$, but not uniformly (B) does not converge at any point (C) converges uniformly on $\mathbb{R}$ (D) converges only at 0.

10. If $1, a_1, a_2, \cdots, a_{n-1}$ are the n^{th} $(n \ge 2)$ roots of unity, then $\frac{1}{1-a_1} + \frac{1}{1-a_2} + \cdots + \frac{1}{1-a_{n-1}} =$
(A) $\frac{n}{2}$ (B) $\frac{n-1}{2}$ (C) $\frac{1}{2}$ (D) $\frac{n(n-1)}{2}$.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $\int_0^1 x(x-1)^2 f(x)dx = 0$. Prove that there is a real number $c \in (0,1)$ such that
$$\int_0^c x^2 f(x)dx = c\int_0^c xf(x)dx.$$

2. Let $f(x) = x^2 + 4x + 3$ and $g(x) = 2x + 5$. Show that $\gcd(f(n), g(n)) = 1$ or 3 as n varies over integers.

3. Let $a_1, a_2, \cdots, a_n$ be real numbers. Prove that $\sum_{1 \le i,j \le n} \frac{a_i a_j}{i+j-1} \ge 0$.

4. Suppose for $n \ge 4$, $x_1, x_2, \cdots, x_n$ are non-negative integers such that after removing any one of them the remaining numbers can be partitioned into two sets with equal sums. (i) If n is even, then prove that $x_i = 0$, for all i. (ii) Does the same conclusion hold when n is odd? Justify your answer.

5. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$, if it exists, which is continuous at a if and only if a is a positive integer. Justify your answer.

Part III

1. Suppose α, β are roots of the polynomial $x^2 + bx + c$. Let $p > 2$ be a prime satisfying the conditions: p divides b, p divides c, but p^2 does not divide c.
(i) Find the largest power of p which divides $\alpha^4 + \beta^4$. [2]
(ii) Find the largest power of p which divides $\alpha^8 + \beta^8$. [3]
(iii) For any positive integer n, find the largest power of p which divides $\alpha^{2^n} + \beta^{2^n}$. [8]

2. (i) Show that $x^6 - 6x^5 + 1 = 0$ has a unique root in the interval $(0,1)$. [3]
(ii) Show that every complex (non-real) root z of $x^6 - 6x^5 + 1 = 0$ satisfies $|z| \le x_0$, where x_0 is the root of this equation in $(0,1)$. [10]

3. (i) Suppose A, B are $n \times n$ complex matrices with $AB^2 - B^2A = B$. Show that B is a nilpotent matrix. [9]
(ii) Let B be an $n \times n$ complex nilpotent matrix. Does there exist an $n \times n$ matrix A such that $AB^2 - B^2A = B$? Justify your answer. [3]

4. Let A, B, C, D be vertices of a quadrilateral inscribed in the unit circle. Let P be a point in the interior of the quadrilateral $ABCD$. Show that
$$|PA||PB||PC||PD| < \frac{27}{16},$$
where $|PX|$ is the length of the line segment PX. [12]

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Solutions and Scheme of marking

N.B.: Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks. Solutions not mentioned here should be graded by analogous schemes.

Part I

N.B. Each question in Part I carries 2 marks.

1. Let $a < b < c$ be three positive integers in geometric progression whose common ratio is also an integer. If the arithmetic mean of a, b, c is $b + 2$, then the value of a is
(A) 24 (B) 12 (C) 6 (D) 4.
Ans: (C)

2. Triangles are formed by joining the vertices of a regular polygon of 12 sides. The number of triangles having no side in common with the polygon is
(A) 124 (B) 220 (C) 96 (D) 112.
Ans: (D)

3. If $f(x) = \begin{cases} ax + b & \text{if } x < -1 \\ x^2 + 1 & \text{if } -1 \leq x < 1 \\ 3x - 1 & \text{if } x \geq 1, \end{cases}$
is continuous everywhere, then which of the following values can ab never take?
(A) -3 (B) -1 (C) 0 (D) 3.
Ans: (A)

4. Let $f : [1, 3] \to \mathbb{R}$ be a continuous function. If f takes rational values for all x and $f(2) = 10$, then $f(1.5) =$
(A) 7.5 (B) 10 (C) 1.5 (D) 5.
Ans: (B)

5. If $f(x) = (x + 1)^2 - 1,\ x \geq -1$ then the set $\{x : f(x) = f^{-1}(x)\}$ is
(A) ϕ (B) $\{-1, 1\}$ (C) $\{0, 1\}$ (D) $\{-1, 0\}$.
Ans: (D)

6. The number of real solutions of $1 + |e^x - 1| = e^x(e^x - 2)$ is
(A) 1 (B) 2 (C) 0 (D) 3.
Ans: (A)

7. In how many ways can you choose an odd number of objects from n objects?
(A) 2^{n-1} (B) 2^n (C) $2^n - 1$ (D) n.
Ans: (A)

8. The family of circles with center $(a, 0)$ and radius a satisfies the differential equation
(A) $\frac{dy}{dx} + x^2 - y^2 = 0$ (B) $2xy\frac{dy}{dx} + x^2 - y^2 = 0$ (C) $2xy\frac{dy}{dx} + x - y = 0$ (D) $2xy\frac{dy}{dx} + x^2 + y^2 = 0$.
Ans: (B)

9. The series $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(x^{2n}) + \sin(x^{2n+1}) - e^{-|x|n}}{1 + n^2}$
(A) converges pointwise everywhere on $\mathbb{R}$, but not uniformly (B) does not converge at any point (C) converges uniformly on $\mathbb{R}$ (D) converges only at 0.
Ans: (C)

10. If $1, a_1, a_2, \cdots, a_{n-1}$ are the n^{th} $(n \geq 2)$ roots of unity, then $\frac{1}{1-a_1}+\frac{1}{1-a_2}+\cdots+\frac{1}{1-a_{n-1}} =$
(A) $\frac{n}{2}$ (B) $\frac{n-1}{2}$ (C) $\frac{1}{2}$ (D) $\frac{n(n-1)}{2}$.
Ans: (B)

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $\int_0^1 x(x-1)^2 f(x)dx = 0$. Prove that there is a real number $c \in (0,1)$ such that

$$\int_0^c x^2 f(x)dx = c\int_0^c xf(x)dx.$$

Solution: Define $g(x) = \int_0^x t(t-x)^2 f(t)dt$. [2]
Observe that $g(0) = 0$ and by given condition, $g(1) = 0$. Therefore, by Rolle's theorem, there exists $c \in (0,1)$ such that $g'(c) = 0$.
Since $g(x) = \int_0^x t(t-x)^2 f(t)dt = \int_0^x t^3 f(t)dt - 2x\int_0^x t^2 f(t)dt + x^2\int_0^x tf(t)dt$,
we get, $g'(x) = x^3 f(x) - 2\int_0^x t^2 f(t)dt - 2x(x^2 f(x)) + 2x\int_0^x tf(t)dt + x^3 f(x)$
$= 2x\int_0^x tf(t)dt - 2\int_0^x t^2 f(t)dt$.
Now $g'(c) = 0$ implies $\int_0^c x^2 f(x)dx = c\int_0^c xf(x)dx$. [4]

2. Let $f(x) = x^2 + 4x + 3$ and $g(x) = 2x + 5$. Show that $\gcd(f(n), g(n)) = 1$ or 3 as n varies over integers.
Solution: Note that $x^2 + 4x + 3 = (2x+5)\left(\frac{1}{2}x + \frac{3}{4}\right) - \frac{3}{4}$. Therefore, we get

$$3 = (2x+5)(2x+3) - 4(x^2+4x+3).$$

Hence $\gcd(f(n), g(n))$ divides 3. Thus, $\gcd(f(n), g(n)) = 1$ or 3. [6]

3. Let $a_1, a_2, \cdots, a_n$ be real numbers. Prove that $\sum_{1\leq i,j\leq n} \frac{a_i a_j}{i+j-1} \geq 0$.
Solution: $0 \leq \int_0^1 \left(\sum_{i=1}^n a_i x^{i-1}\right)^2 dx = \int_0^1 \sum_{1\leq i,j\leq n} a_i a_j x^{i+j-2} dx$
$= \sum_{1\leq i,j\leq n} a_i a_j \int_0^1 x^{i+j-2}dx = \sum_{1\leq i,j\leq n} \frac{a_i a_j}{i+j-1}$. [6]

4. Suppose for $n \geq 4$, $x_1, x_2, \cdots, x_n$ are non-negative integers such that after removing any one of them the remaining numbers can be partitioned into two sets with equal sums. (i) If n is even, then prove that $x_i = 0$, for all i. (ii) Does the same conclusion hold when n is odd? Justify your answer.
Solution: (i) If n is even, then all x_i must be even. Indeed, if any x_i is odd, then there exist two cases: (a) Suppose the number of odd x_i is odd. Then taking an even number out, the parity of the two sets will not match. (b) If the number of odd x_i is even, then taking an odd number out, the parity of the two sets will not match.
Let $y_i = x_i/2$. Then the new set $y_1, y_2, \cdots, y_n$ also has the same property. Hence all y_i are even. Continuing this way we will reach at at least one odd number in the set that gives the contradiction. Hence $x_i = 0$, for all i. [4]
(ii) NO. A counter example is $\{1,1,1,1,1,1,1\}$ **OR** any other counter example. [2]

5. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$, if it exists, which is continuous at a if and only if a is a positive integer. Justify your answer.
Solution: Define a function $f : \mathbb{R} \to \mathbb{R}$, as follows:
$f(x) = \begin{cases} h(x) & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$
where $h(x) = \begin{cases} 1 & \text{if } x \leq 1/2 \\ \sin(\pi x) & \text{if } x > 1/2. \end{cases}$
Note that $h(x) = 0$ for all positive integers x. Hence the function $f(x)$ is continuous at a if and only if a is a positive integer.
OR Any other correct example. [6]

Part III

1. Suppose α, β are roots of the polynomial $x^2 + bx + c$. Let $p > 2$ be a prime satisfying the conditions: p divides b, p divides c, but p^2 does not divide c.
(i) Find the largest power of p which divides $\alpha^4 + \beta^4$. [2]
(ii) Find the largest power of p which divides $\alpha^8 + \beta^8$. [3]
(iii) For any positive integer n, find the largest power of p which divides $\alpha^{2^n} + \beta^{2^n}$. [8]
Solution: Define $a_n = \alpha^{2^n} + \beta^{2^n}$, for all $n \in \mathbb{N}$.
Then $a_1 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = b^2 - 2c$. Since p divides b and c, p divides a_1. Since p^2 divides b^2 but it does not divide c, p^2 does not divide a_1. Hence p^1 the largest power of p which divides a_1.
(i) Let $a_2 = \alpha^4 + \beta^4 = a_1^2 - 2\alpha^2\beta^2 = (b^2 - 2c)^2 - 2c^2$. This implies that p^2 divides a_2. Claim: p^3 does not divide a_2. Observe that $a_2 = (b^2 - 2c)^2 - 2c^2 = b^4 - 4b^2c + 2c^2$. Since p^3 does not divide c^2, but it divides other terms, we have p^3 does not divide a_2. Hence p^2 is the largest power of p which divides a_2. [2]
(ii) Let $a_3 = \alpha^8 + \beta^8 = a_2^2 - 2\alpha^4\beta^4 = a_2^2 - 2c^4 = (b^4 - 4b^2c + 2c^2)^2 - 2c^4 = b^8 - 8b^6c + 20b^4c^2 - 16b^2c^3 + 2c^4$. Now by case (i), p^4 divides a_3. Since p^5 divides all other terms but it does not divide c^4, we get p^5 does not divide a_3. Hence p^4 is the largest power of p which divides a_3. [3]
(iii) Claim: For all $n \geq 2$, there exists an integer b_n such that $a_n = p^{2^{n-1}+1}b_n + 2c^{2^{n-1}}$. Proof of the claim is by induction on n. For $n = 2$, we have $a_2 = p^3b_2 + 2c^2$. Assume that the result is true for some positive integer n. Therefore $a_n = p^{2^{n-1}+1}b_n + 2c^{2^{n-1}}$. Now $a_{n+1} = a_n^2 - 2\alpha^{2^n}\beta^{2^n} = a_n^2 - 2c^{2^n} = (p^{2^{n-1}+1}b_n + 2c^{2^{n-1}})^2 - 2c^{2^n} = p^{2^n+1}b_{n+1} + 2c^{2^n}$. Hence the claim is true for $n + 1$. Thus, the claim is true for all $n \geq 2$. Since $p^{2^{n-1}}$ divides $c^{2^{n-1}}$, we have $p^{2^{n-1}}$ divides a_n. Since $p^{2^{n-1}+1}$ does not divide $c^{2^{n-1}}$ we get, $p^{2^{n-1}+1}$ does not divide a_n. Hence $p^{2^{n-1}}$ is the largest power of p which divides a_n. [8]

2. (i) Show that $x^6 - 6x^5 + 1 = 0$ has a unique root in the interval $(0, 1)$. [3]
(ii) Show that every complex (non-real) root z of $x^6 - 6x^5 + 1 = 0$ satisfies $|z| \leq x_0$, where x_0 is the root of this equation in $(0, 1)$. [10]
Solution: (i) Let $f(x) = x^6 - 6x^5 + 1$ for $x \geq 0$. Observe that $f(0) = 1 > 0$ and $f(1) = -4 < 0$. Therefore there exists a root in the interval $(0, 1)$. [1]
Also $f(6) = 1 > 0$. Therefore there exists a root in the interval $(1, 6)$. Since there are only two sign changes, by Descarte's rule f has at most two positive real roots. Hence f has a unique root in the interval $(0, 1)$. [2]
OR Note that $f'(x) = 6x^5 - 30x^4 = 6x^4(x - 5) \leq 0$ for $0 \leq x \leq 5$ and $f'(x) \geq 0$ for $x \geq 5$. Therefore, f is decreasing for $0 \leq x \leq 5$ and increasing for $x \geq 5$. Hence f has a unique root in the interval $(0, 1)$. [2]
Note: Correct argument for uniqueness will get 2 marks.
Method 1:
(ii) Let x_0 be the unique root of f in the interval $(0, 1)$. Since $f(0) > 0$, we have $f(x) > 0$ for $0 \leq x < x_0$. Let $z = re^{i\theta}$ be a complex root of f where $r > 0$ and $-\pi < \theta < \pi, \theta \neq 0$.

The real and imaginary parts of the equation $f(x) = 0$ gives

$$r^6 \cos 6\theta - 6r^5 \cos 5\theta = -1$$

$$r^6 \sin 6\theta - 6r^5 \sin 5\theta = 0.$$

Solving these equations we get,
$r^6 = \dfrac{\sin 5\theta}{\sin \theta}$, $r^5 = \dfrac{\sin 6\theta}{6 \sin \theta}$. [$*$] [4]
It can be seen that $r^6 < 5$. Hence we have $r < 5$. To show that $r < x_0$, we must show that $f(r) > 0$.

$$f(r) = r^6 - 6r^5 + 1 = \frac{\sin 5\theta}{\sin \theta} - 6\frac{\sin 6\theta}{6 \sin \theta} + 1 = \frac{\cos \frac{\theta}{2} - \cos \frac{11\theta}{2}}{\cos \frac{\theta}{2}}.$$

Note that $\cos \dfrac{\theta}{2} > 0$. If $\cos \dfrac{11\theta}{2} \leq 0$, then $f(r) \geq 1 > 0$.
If $\cos \dfrac{11\theta}{2} > 0$, then the first factor in the following is positive:
$\left(\cos \dfrac{\theta}{2} + \cos \dfrac{11\theta}{2}\right)\left(\cos \dfrac{\theta}{2} - \cos \dfrac{11\theta}{2}\right) = \dfrac{\cos \theta - \cos 11\theta}{2} = \sin 5\theta \sin 6\theta.$
Right hand side of the above expression is positive by [$*$]. Hence second factor is positive. Thus $f(r) > 0$. Since $r < 5$ and $f(r) > 0$, we get $r < x_0$. [6]
OR Method 2: (ii) Let x_1 be the root of f in the interval $(1, 6)$. If $x_0 < x < 1$, then $f(x) < 0$. Therefore $|z^6 + 1| \leq |z^6| + 1 < |6z^5|$ if $|z| = x$. So by Rouche's theorem, $z^6 - 6z^5 + 1$ also has 5 zeros inside $|z| = x$. Since this is true for every $x > x_0$ all zeros except x_1 satisfy $|z| \leq x_0$. [10]

3. (i) Suppose A, B are $n \times n$ complex matrices with $AB^2 - B^2A = B$. Show that B is a nilpotent matrix. [9]
(ii) Let B be an $n \times n$ complex nilpotent matrix. Does there exist an $n \times n$ matrix A such that $AB^2 - B^2A = B$? Justify your answer. [3]
Solution: (i) Claim: $AB^{2^n} - B^{2^n}A = 2^{n-1}B^{2^n-1}$, for all $n \geq 2$. The proof is by induction on n. We first prove the result for $n = 2$. Given that

$$AB^2 - B^2A = B.$$

Multiplying both sides by B^2 from right we get, $AB^4 - B^2AB^2 = B^3$. Substituting $AB^2 = B + B^2A$, we get $AB^4 - B^2(B + B^2A) = B^3$. Therefore

$$AB^4 - B^4A = 2B^3.$$

Hence, the result is true for $n = 2$. Suppose the result is true for some positive integer $k > 2$. Therefore,

$$AB^{2^k} - B^{2^k}A = 2^{k-1}B^{2^k-1}.$$

Multiplying both sides by B^{2^k} from right we get,

$$AB^{2^k}B^{2^k} - B^{2^k}AB^{2^k} = 2^{k-1}B^{2^k-1}B^{2^k}.$$

Then put $AB^{2^k} = 2^{k-1}B^{2^k-1} + B^{2^k}A$. Hence

$$AB^{2^{k+1}} - B^{2^{k+1}}A = 2^kB^{2^{k+1}-1}.$$

Therefore the result is true for $k + 1$. Thus, the result is true for all $n \geq 2$. [6]
By taking norm we get,

$$2^{n-1}||B^{2^n-1}|| \leq 2||A||||B^{2^n}||.$$

Suppose B is not a nilpotent matrix. Then $||B^k|| \neq 0,\ \forall k \geq 1$. After cancellation we get, $2^{n-1} \leq 2||A||.||B||$ for all positive integers $n \geq 2$. This is contradiction. [3]
(ii) Let $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $B^2 = 0$. Therefore there does not exist a matrix A such that $AB^2 - B^2A = B$. **OR** Any other correct example. [3]

4. Let A, B, C, D be vertices of a quadrilateral inscribed in the unit circle. Let P be a point in the interior of the quadrilateral $ABCD$. Show that

$$|PA||PB||PC||PD| < \frac{27}{16},$$

where $|PX|$ is the length of the line segment PX. [12]
Solution: Suppose P is a point on the boundary of the quadrilateral $ABCD$ say on BC. Let x be the distance of P from the center of the unit circle O. We know that for all chords XY through point P the product $|PX||PY|$ is constant. Therefore

$$|PB||PC| = (1 - x)(1 + x).$$

[6]
Applying triangle inequality to triangle OPA, $|PA| < 1 + x$. Similarly $|PD| < 1 + x$. Hence

$$|PA||PB||PC||PD| < (1 - x)(1 + x)^3.$$

Let $f(x) = (1 - x)(1 + x)^3$. Then $f'(x) = (1 + x)^2(2 - 4x) = 0$ at $x = \frac{1}{2}$. Observe that $f''(1/2) < 0$. Now $f\left(\frac{1}{2}\right) = \frac{27}{16}$ is the maximum value. By Maximum Modulus Principle, the maximum value cannot occur at any interior point.
Hence, $|PA||PB||PC||PD| < \frac{27}{16}$. [6]